Copyright Warning

Use of this thesis/dissertation/project is for the purpose of private study or scholarly research only. Users must comply with the Copyright Ordinance.

Anyone who consults this thesis/dissertation/project is understood to recognise that its copyright rests with its author and that no part of it may be reproduced without the author’s prior written consent.
CITY UNIVERSITY OF HONG KONG
香港城市大學

p-Version Finite Elements
在結構動態穩定性中的高階譜有限元
in Structural Dynamics and Stability

Submitted to
Department of Building and Construction
建築系
in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy
哲學博士學位

by

Fan Jie
范潔

September 2010
二零一零年九月
ABSTRACT

The performance of the finite element method (FEM) may be improved in three ways. The first is by the $h$-version to refine the finite element mesh and the second is by the $p$-version to increase the order of polynomial shape functions for a fixed mesh. The third is simultaneously to refine the mesh and increase the degrees of elements uniformly or selectively, called the $h$-$p$ version, which is a combination of the first two methods. The advantages of the $p$-version elements over the $h$-version are: (i) they have better conditioned matrices; (ii) they do not require a change in the mesh and can be easily used in the adaptive analysis; (iii) just one element can predict accurate solutions for a simple structure; (iv) they tend to give the same accurate results with far fewer degrees of freedom (DOF); and (v) they can overcome some locking problems.

The main objectives of present study are to give a wide range of application of the Fourier $p$-elements and polynomial $p$-elements to investigate the vibration problems, buckling problems and dynamic stability problems of various conservative linear structures including beam-columns, straight Timoshenko beams, pre-twisted straight beams, Mindlin plates and open thin shell panels. The natural frequencies, buckling loads and the relation of frequencies and various buckling loads are considered. Good agreement is achieved with the available results.

The main work can be divided into five parts. Firstly, the numerical examples of uniform or tapered beam-columns with or without end mass are considered and compared with results of dynamic stiffness method. New results of straight beam-columns subjected to uniformly distributed follower tension are originally
reported. Secondly, the axial-torsional buckling of space straight beams based on unequally shared end torque theory is studied. Then flexural-torsional buckling problems of space straight beams subject to end moments, end shear loads and distributed shear loads are investigated. Thirdly, the effects of pre-twist rate and rigidity ratio on dynamic stability of pre-twisted straight beams are given. The natural frequencies and buckling loads of pre-twisted beams subject to axial loads, torque, moments and shears are discussed in detail. Fourthly, dynamic stability problems of Mindlin plates with rectangular, skew, trapezoidal, triangular, polygonal shapes are analyzed. Problems of plate systems composed of rectangular and/or trapezoidal elements with different thicknesses are discussed. Finally, the influence of aspect ratio, circumferential angles on the natural frequencies and vibration mode shapes of open thin cylindrical, conical and spherical shell panels are studied. The buckling problems of cylindrical shell panels under axially compressed loads are finally investigated.
ACKNOWLEDGEMENT

The author wishes to express her sincere gratitude to her supervisor, Prof. Andrew Y. T. Leung, for his sustained encouragement, interest and guidance throughout the period of her research. It has been extremely rewarding and fruitful for the author to work with Prof. Andrew Y. T. Leung. With his preeminent research keenness and practical experience as a scholar, the author always obtained timely and valuable directions to overcome difficulties met in the PhD study and research. The author would like to express her great honor and pleasure in her life to have the chance to study with him.

The author is also most appreciative of her family for their constant encouragement and supports.

The author would like to thank Dr. Raymond Y. Y. Lee, Dr. Walter H. Y. Sun and Dr. Y. D. Wu for their kindly help on the research.

Grateful thanks are expressed to all friends of the author in City University of Hong Kong and Huazhong University of Science and Technology, P. R. China.

Last but not least, special thanks to Prof. W. F. Zhong, the supervisor throughout the period of the author’s MPhil study, for his kindly encouragement and recommendation to obtain the chance to work with Prof. Andrew Y. T. Leung and to study in City University of Hong Kong.
CHAPTER 1 INTRODUCTION-----------------------------------------------1

1.1 Introduction to the \( p \)-element method................................1
1.2 Brief view of the application of \( p \)-element method.................3
1.3 Scope of present study.......................................................4
1.4 General formulation of \( p \)-element......................................6

1.4.1 Shape functions.........................................................6
1.4.2 Mapping functions......................................................9
1.4.3 Integration procedures.................................................10

CHAPTER 2 FOURIER \( p \)-ELEMENTS FOR DYNAMIC STABILITY
OF COLUMNS-----------------------------------------------------------11

2.1 Introduction.................................................................12

2.1.1 Condition numbers of Fourier \( p \)-elements and polynomial
\( p \)-elements.................................................................12
2.1.2 The concept of follower forces ..................................................13
2.2 Governing equations of beam-columns ........................................15
2.3 Fourier $p$-elements for beam-columns .......................................16
   2.3.1 Shape functions ...............................................................16
   2.3.2 Stiffness, mass and geometric stiffness matrices ......................17
   2.3.3 Formulation of follower forces ............................................18
2.4 Numerical examples .................................................................21
   2.4.1 Buckling problems of beams under conservative concentrated
         loads ....................................................................................22
   2.4.2 High modes comparison for buckling problems of beams under
         follower forces .................................................................26
   2.4.3 Buckling problems of beams under follower forces .................28
   2.4.4 Interaction diagram of beams under conservative tension and
         follower tension ...............................................................30
   2.4.5 Interaction diagram of tapered beams ....................................35
   2.4.6 Interaction diagram of columns with end mass .........................40
   2.4.7 Interaction diagram of columns subject to combined concentrated
         follower tension and distributed follower tension ......................43
2.5 Conclusion .................................................................................46

CHAPTER 3 FOURIER $p$-ELEMENTS FOR DYNAMIC STABILITY

OF SPACE BEAMS ........................................................................47
3.1 Introduction ................................................................................48
3.2 Description of the problems and governing equation of space beams...49
3.3 Formulation of Fourier $p$-element for space beams ......................51
   3.3.1 Shape functions ..................................................................51
   3.3.2 Stiffness, mass and geometric stiffness matrices ......................52
3.4  Numerical examples for axial-torsional buckling of space beams……..55
   3.4.1  Buckling problems of uniform beams.................................55
   3.4.2  Buckling problems of two section step beam..........................64
3.5  Numerical examples of flexural-torsional buckling of space beams......67
3.6  Conclusion..................................................................................74

CHAPTER 4 POLYNOMIAL $p$-ELEMENTS FOR DYNAMIC STABILITY
   OF PRE-TWISTED STRAIGHT BEAMS---------------------75
4.1  Introduction..................................................................................76
4.2  Incremental strain analysis..........................................................78
4.3  Geometry......................................................................................81
   4.3.1  The geometry of a general curve with a pre-twist rate.................81
   4.3.2  Displacement, strain and external force.....................................84
4.4  Energy...........................................................................................88
4.5  $p$-elements for pre-twisted straight beams....................................92
   4.5.1  Shape functions........................................................................92
   4.5.2  Stiffness, mass and geometric stiffness matrices..........................93
4.6  Numerical examples........................................................................96
   4.6.1  Results comparison.................................................................96
   4.6.2  The influence of rigidity ratio and pre-twisted angle...................99
   4.6.3  Interaction of natural frequency and buckling loads.................106
4.7  Conclusion....................................................................................117

CHAPTER 5 FOURIER $p$-ELEMENTS FOR DYNAMIC STABILITY
   OF MINDLIN PLATES---------------------------------------------118
5.1  Introduction..................................................................................119
5.2  Governing equations of Mindlin plates..........................................121
5.3 Fourier $p$-elements for Mindlin plates……………………………………123
  5.3.1 Shape functions………………………………………………………123
  5.3.2 Mapping…………………………………………………………………124
  5.3.3 Stiffness, mass and geometric stiffness matrices…………………125
5.4 Numerical examples………………………………………………………126
  5.4.1 Square plates and skew plates……………………………………127
  5.4.2 Trapezoidal plates…………………………………………………134
  5.4.3 Polygonal plates…………………………………………………136
  5.4.4 Plate systems with different thickness plate elements……………145
5.5 Conclusion………………………………………………………………152

CHAPTER 6 POLYNOMIAL $p$-ELEMENTS FOR VIBRATION AND
BUCKLING OF OPEN, THIN SHELL PANELS———153
6.1 Introduction………………………………………………………………154
6.2 Governing equations……………………………………………………156
6.3 $p$-elements for shear deformable shell panels………………………169
  6.3.1 Shape functions……………………………………………………169
  6.3.2 Mapping……………………………………………………………170
  6.3.3 Stiffness, mass and geometric stiffness matrices…………………172
  6.3.4 Integration procedures…………………………………………173
6.4 Natural frequency of shell panels………………………………………174
  6.4.1 Convergence study and comparison with existed solutions………174
  6.4.2 Vibration problems of cylindrical shell panels…………………..176
  6.4.3 Vibration problems of conical shell panels………………………177
  6.4.4 Vibration problems of spherical shell panels……………………178
  6.4.5 Vibration mode shapes of open cylindrical, conical and spherical
      shell panels……………………………………………………………179
6.5 Buckling problems of open cylindrical shell panels..................180
   6.5.1 Convergence study and comparison with results of ANSYS.......180
   6.5.2 Buckling loads of open cylindrical shell panels................181
6.6 Vibration problems of shell panels with large circumferential angles or
   thinner thickness.............................................................182
6.7 Conclusion........................................................................199

CHAPTER 7 CONCLUSION AND RECOMMENDATIONS----------------200
  7.1 Conclusion and limitations of the research..........................200
  7.2 Recommendation for further research.................................202

BIBLIOGRAPHY----------------------------------------------------------204
APPENDIX 1-------------------------------------------------------------226
APPENDIX 2-------------------------------------------------------------229
APPENDIX 3-------------------------------------------------------------236
APPENDIX 4-------------------------------------------------------------246
APPENDIX 5-------------------------------------------------------------250
APPENDIX 6-------------------------------------------------------------251
APPENDIX 7-------------------------------------------------------------254
APPENDIX 8-------------------------------------------------------------257
VITA-----------------------------------------------------------------------261
LIST OF PUBLICATIONS-----------------------------------------------262
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>DOF and type of shape functions used in this thesis</td>
</tr>
<tr>
<td>3.1</td>
<td>Buckling axial compressive load parameters $P^{1/2}$ comparison</td>
</tr>
<tr>
<td>3.2</td>
<td>Natural frequency parameters $\Omega_p^{1/4}$ comparison</td>
</tr>
<tr>
<td>3.3</td>
<td>Buckling load parameters $M^{1/2}$ and $N^{1/2}$ comparison</td>
</tr>
<tr>
<td>3.4</td>
<td>Frequency and buckling load parameters $\Omega_{X'}^{1/2}$, $\Omega_{Y'}^{1/2}$, $Q^{1/2}$ and $q^{1/2}$ of the I-section beam</td>
</tr>
<tr>
<td>4.1</td>
<td>Convergence of natural frequencies with comparison to existing results (rad/s)</td>
</tr>
<tr>
<td>4.2</td>
<td>Frequency $\Omega$ and buckling loads load parameters $\sqrt{q_1}$, $\sqrt{q_2}$, $\sqrt{q_3}$ of a pre-twisted clamped-free beam with $l=1$m, $b=1/1.5$mm, $d=1.5$mm, $\overline{\mu} = \pi/2$ and compared with results of ANSYS</td>
</tr>
<tr>
<td>4.3</td>
<td>The relative error when taking ($J \neq J_0$) and ($J=J_0$) for frequency $\Omega$ and buckling loads parameters $\sqrt{m_1}$, $\sqrt{m_2}$, $\sqrt{m_3}$, $\sqrt{m_1}$, $\sqrt{m_2}$ and $m_3$ of a pre-twisted clamped-free beam with $l=1$m, $b=1/1.5$mm, $d=1.5$mm, $\overline{\mu} = \pi/2$</td>
</tr>
<tr>
<td>4.4</td>
<td>The flexural rigidity ratio $r$ when the width $d$ varying from 1 to 1.4 ($i=1$~5)</td>
</tr>
<tr>
<td>4.5a</td>
<td>Influence of pre-twist angle $\overline{\mu}$ and rigidity ratio $r$ on natural frequency</td>
</tr>
</tbody>
</table>
4.5b Influence of pre-twist angle $\bar{\mu}$ and rigidity ratio $r$ on buckling loads $\sqrt{q_3}$ .................................................................101

4.5c Influence of pre-twist angle $\bar{\mu}$ and rigidity ratio $r$ on buckling loads $\sqrt{q_1}$ .................................................................101

4.5d Influence of pre-twist angle $\bar{\mu}$ and rigidity ratio $r$ on buckling loads $\sqrt{q_2}$ .................................................................102

4.5e Influence of pre-twist angle $\bar{\mu}$ and rigidity ratio $r$ on buckling loads $\sqrt{m_1}$ .................................................................102

4.5f Influence of pre-twist angle $\bar{\mu}$ and rigidity ratio $r$ on buckling loads $\sqrt{m_2}$ .................................................................102

4.5g Influence of pre-twist angle $\bar{\mu}$ and rigidity ratio $r$ on buckling loads $m_3(-)$ .................................................................103

4.5h Influence of pre-twist angle $\bar{\mu}$ and rigidity ratio $r$ on buckling loads $m_3(+)$. .................................................................103

5.1 Natural frequency parameters $\Omega = \omega L_p^2 \sqrt{\rho t / D} (L_p/t=10, k=\pi^2/12, v=0.3)$ for a CFF equilateral triangular plate ........................................139

5.2 Natural frequency parameters $\Omega = \omega L_p^2 \sqrt{\rho t / D / \pi^2}$ and buckling load intensity factors $\lambda = TL_p^2 / (\pi^2 D)$ for a SSS and CCC equilateral triangular plate ($L_p/t=10, k=5/6, v=0.3$) .................................................................139

5.3 Natural frequency parameters $\Omega = \omega L_p^2 \sqrt{\rho t / D / \pi^2}$ and buckling load intensity factors $\lambda = TL_p^2 / (\pi^2 D)$ for a SSSS and CCCC square plate
5.4 Natural frequency parameters $\Omega = \omega L_p^2 \sqrt{\rho t / D / \pi^2}$ and buckling load intensity factors $\lambda = T L_p^2 / \left( \pi^2 D \right)$ for a SSSSSS and CCCCCC hexagonal plate ($L_p/t=10$, $k=5/6$, $\nu=0.3$)……………………………………………………………..140

5.5 Natural frequency parameters $\Omega = \omega L_p^2 \sqrt{\rho t / D / \pi^2}$ and buckling load intensity factors $\lambda = T L_p^2 / \left( \pi^2 D \right)$ for a SSSSSSS and CCCCCCCC octagonal plate ($L_p/t=10$, $k=5/6$, $\nu=0.3$)……………………………………………………………..141

5.6 Natural frequency parameters $\Omega = \omega L_p^2 \sqrt{\rho t / D / \pi^2}$ and buckling load intensity factors $\lambda = T L_p^2 / \left( \pi^2 D \right)$ for a FFFC square plate composed of two plate elements with different thickness $t_1$ ($L_p/t_2=10$, $k=5/6$, $\nu=0.3$)……………………………………………………………..147

5.7 Natural frequency parameters $\Omega = \omega L_p^2 \sqrt{\rho t / D / \pi^2}$ and buckling load intensity factors $\lambda = T L_p^2 / \left( \pi^2 D \right)$ for a CCCC square plate system composed of four plate elements with different thickness $t_1$ ($L_p/t_2=10$, $k=5/6$, $\nu=0.3$)……………………………………………………………..150

6.1 Frequency parameters $\Omega=\omega(L_\delta)^2(\rho t/D)^{1/2}$ convergence comparison for CCCC cylindrical shallow shell panels with thickness ratio $t/L_\delta=0.01$ and 0.05…………………………………………………………………………………..175

6.2 Frequency convergence and comparison of $\Omega = \omega L_p^2 \sqrt{\rho t / D}$ for a fully clamped thin conical shell panel with $\nu=0.3$, $s_\delta/L_\delta=2/3$, $t/L_\delta=0.01$, $\psi=\pi/6$ and $L_f=\pi/3$…………………………………………………………………………………..176

6.3 Convergence and comparison of $\Omega = \omega b \sqrt{\rho / E}$ for a fully clamped shallow spherical shell panels with square planform of width $b$, and $t/b=0.01$, $a/b=2$, $f_0=\arccos(0.5b/a)$, $L_f=2\arcsin(0.5b/a)$,
\[ L_s = 2 \arcsin \left( \frac{0.5b}{a} \right) \]………………………………………………………176

6.4 Frequency parameters \( \Omega = \alpha \sqrt{\frac{\rho}{E}} \) for fully clamped, thin, open cylindrical shell panels with \( a=1m \) and \( t/a=0.01 \)………………………………………………………184

6.5a Frequency parameters \( \Omega = \alpha \sqrt{\frac{\rho}{E}} \) for fully clamped, thin, open conical shell panels with \( \psi = \pi/6, s_0 = 0.5m \) and \( t=0.01m \)………………………………………………………185

6.5b Frequency parameters \( \Omega = \alpha \sqrt{\frac{\rho}{E}} \) for fully clamped, thin, open conical shell panels with \( \psi = \pi/6, s_0 = 1m \) and \( t=0.01m \)………………………………………………………186

6.6a Frequency parameters \( \Omega = \alpha \sqrt{\frac{\rho}{E}} \) for fully clamped, thin, open conical shell panels with \( \psi = \pi/3, s_0 = 0.5m \) and \( t=0.01m \)………………………………………………………187

6.6b Frequency parameters \( \Omega = \alpha \sqrt{\frac{\rho}{E}} \) for fully clamped, thin, open conical shell panels with \( \psi = \pi/3, s_0 = 1m \) and \( t=0.01m \)………………………………………………………188

6.7 Frequency parameters \( \Omega = \alpha \sqrt{\frac{\rho}{E}} \) for fully clamped, thin, open spherical shell panels with \( a=1m, f_0 = \pi/6, \) and \( t/a=0.01 \)………………………………………………………189

6.8 Frequency parameters \( \Omega = \alpha \sqrt{\frac{\rho}{E}} \) for fully clamped, thin, open spherical shell panels with \( a=1m, f_0 = \pi/4 \) and \( t/a=0.01 \)………………………………………………………190

6.9a Mode shapes comparison with ANSYS for a fully clamped, thin, open cylindrical shell panel with \( a=1m, L_s/a=1, L_f=\pi/2 \) and \( t/a=0.01 \)………………..191

6.9b Mode shape comparison for a fully clamped, thin, open conical shell panel with \( L_s=1m, s_0/L_s=1, t/L_s=0.01, \psi = \pi/6 \) and \( L_f=\pi/2 \)………………………………………………………192

6.9c Mode shape comparison with ANSYS for a fully clamped, thin, open spherical shell panel with \( a=1m, t/a=0.01, f_0=\pi/6, L_f=\pi/3 \) and \( L_s=\pi/2 \)…193

6.10a Convergence and comparison of buckling load intensity factors 
\[ \lambda = N_c a \sqrt{3 \left(1-v^2\right)} \left(\frac{1}{Et^2}\right) \] for a CFFF cylindrical shell panel ………..181

6.10b Buckling mode shape comparison of a CFFF cylindrical shell panel with results of ANSYS……………………………………………………………………………181
6.11 Buckling load intensity factors $\lambda = N_c a \sqrt{\frac{3(1-v^2)}{E t}}$ for thin, open cylindrical shell panels with $a=1\text{m}$, $t/a=0.01$ with CFFF and CCCC boundary conditions.................................................................194

6.12 Frequency parameters $\Omega = a\alpha \sqrt{\rho E}$ for fully clamped, thin, open cylindrical shell panels with $a=1\text{m}$, $L_f=\pi/2$ or $\pi$, $t/a=0.01$ or 0.001.................................195

6.13 Frequency parameters $\Omega = a\alpha \sqrt{\rho E}$ for fully clamped, thin, open conical shell panels with $L_s=1\text{m}$, $\psi=\pi/4$, $s_0/(L_s+s_0)=0.4$, $L_f=\pi/2$ or $\pi$, $t/L_s=0.01$ or 0.001.................................................................196

6.14 Frequency parameters $\Omega = a\alpha \sqrt{\rho E}$ for fully clamped, thin, open spherical shell panels with $a=1\text{m}$, $f_0=\pi/4$, $L_f=\pi/4$, $L_s=\pi/2$ or $\pi$, $t/a=0.01$ or 0.001...198
LIST OF FIGURES

2.1 Condition numbers.................................................................13
2.2 A plane bending beam subject to axial force $P$.........................15
2.3 Columns subject to conservative forces or follower forces..........19
2.4a Clamped- Free beam subject to conservative axial compression......22
2.4b Hinged-Hinged beam subject to conservative axial compression....23
2.4c Clamped- Hinged beam subject to conservative axial compression...23
2.4d Clamped- Clamped beam subject to conservative axial compression.24
2.5a Clamped- Free beam subject to conservative axial tension............25
2.5b Hinged- Hinged beam subject to conservative axial tension.........25
2.5c Clamped- Hinged beam subject to conservative axial tension........26
2.5d Clamped- Clamped beam subject to conservative axial tension.......26
2.6a High modes of Clamped-Free beam subject to concentrated follower compression.................................................................27
2.6b High modes of Clamped-Free beam subject to concentrated follower tension..................................................................................28
2.7 Uniform column subject to concentrated follower tension..........29
2.8 Eigenvalue curve $g=g(\omega^2)$ of a column subject to uniformly distributed follower compression by Fourier $p$-element.................30
2.9a Clamped-Free column subject to concentrated follower tension....31
2.9b Hinged-Hinged column subject to concentrated follower tension....31
2.9c Clamped-Hinged column subject to concentrated follower tension...32
2.9d  Clamped-Clamped column subject to concentrated follower tension…32
2.10a  Clamped-Free column subject to distributed follower tension………33
2.10b  Hinged-Hinged column subject to distributed follower tension………34
2.10c  Clamped-Hinged column subject to distributed follower tension……34
2.10d  Clamped-Clamped column subject to distributed follower tension……35
2.11  Enlarged portions at the first mode under uniformly distributed……33
2.12  A vessel loaded a liquid…………………………………………………35
2.13  Uniform beam and tapered beam used in Section 2.4.5………………36
2.14a  Uniform beam and tapered beam under conservative axial compression…………………………………………………………37
2.14b  Uniform beam and tapered beam under follower concentrated compression…………………………………………………………38
2.14c  Uniform column and tapered column subject to distributed follower compression…………………………………………………………38
2.14d  Uniform beam and tapered beam under conservative axial tension……39
2.14e  Uniform and tapered columns under concentrated follower tension……39
2.14f  Uniform column and tapered column subject to distributed follower tension…………………………………………………………40
2.15  Clamped-Free column subject to follower tension with end mass……41
2.16a  Column subject to concentrated follower tension with or without end mass………………………………………………………………41
2.16b  Column subject to distributed follower tension with or without end mass………………………………………………………………42
2.16c  Column subject to concentrated follower compression with or without end mass…………………………………………………………42
2.16d  Column subject to distributed follower compression with or without end mass…………………………………………………………43
2.17a 3D plot of $\left( \Omega, \sigma_q, \sigma_p \right)$ for column subject to combined follower tension.................................................................44

2.17b Contour plot for $\left( \Omega, \sigma_q, \sigma_p \right)$...............................................................44

2.18a 3D plot of $\left( \Omega, \sigma_p, \sigma_q \right)$ for column subject to combined follower tension.................................................................45

2.18b Contour plot for $\left( \Omega, \sigma_p, \sigma_q \right)$...............................................................45

3.1 A space beam element subject to axial tension $P$, torque $L$ and end moment $M$ and $N$.................................................................49

3.2 A cantilevered beam subject to axial loads and torque.........................55

3.3a Interaction diagram between buckling compression and natural frequency......................................................................................57

3.3b Interaction diagram between buckling tension and natural frequency......................................................................................58

3.3c Interaction diagram between buckling tension and natural frequency..................58

3.4a The influence of the buckling torque on axial compression...............59

3.4b The influence of the axial compression on buckling torque...............60

3.4c The influence of the buckling torque on axial tension.........................61

3.4d The influence of the axial tension on buckling torque.........................61

3.5a The interaction diagram between the buckling torque and compression.62

3.5b The interaction diagram between the buckling torque and tension...........62

3.6a 3D interaction diagram for axial compression........................................63

3.6b 3D interaction diagram for axial tension...............................................63

3.7 A cantilevered two-section step beam.........................................................64

3.8a Interaction diagram between axial compression and natural frequency of the two section beam.........................................................65

3.8b Interaction diagram between axial tension and natural frequency of the two section beam.........................................................65
3.8c Interaction diagram between buckling torque and natural frequency of the two section beam…………………………………………………………66
3.9 The influence of buckling torque on axial compression for the two section beam………………………………………………………………………66
3.10 3D interaction diagram for the two section beam……………………………………...…………….67
3.11 Cantilevers subject to end moment $M$, end load $Q$, or uniform distributed loads $q$………………………………………………………………………………..68
3.12 An I-section space beam…………………………………………………………..68
3.13 Interaction diagram between natural frequency $\Omega^{1/4}$ and end moment $M^{1/2}$………………………………………………………………………………..71
3.14 Interaction diagram between natural frequency $\Omega^{1/4}$ and end moment $N^{1/2}$………………………………………………………………………………..72
3.15 Interaction diagram between natural frequency $\Omega^{1/4}$ and end load $Q^{1/2}$………………………………………………………………………………..72
3.16 Interaction diagram between natural frequency $\Omega^{1/4}$ and distributed load $q^{1/2}$………………………………………………………………………………..73
3.17 The relation between end moment $M^{1/2}$ and end moment $N^{1/2}$, end load $Q^{1/2}$ or distributed load $q^{1/2}$…………………………………………………………..73
4.1 A pre-twisted straight beam…………………………………………………………..83
4.2 Displacements and external forces of a straight beam………………………………………………………………………………………………………………..84
4.3 The dimensions of the cantilevered beam………………………………………………………………………………………………………………………………97
4.4 Natural frequencies and buckling loads under the influence of initial twist angle and flexural rigidity ratio…………………………………………………………104
4.5 The influence of the twist angle on interaction diagram of natural frequencies and buckling compressions…………………………………………………………107
4.6 The influence of the angle of twist on interaction diagram of natural frequencies and buckling torques $m_3(-)$…………………………………………………………108
4.7 The influence of the angle of twist on interaction diagram of natural frequencies and buckling torques $m_3(\pm)$……………………………109
4.8 The influence of the angle of twist on interaction diagram of natural frequency and shear buckling loads $\sqrt{q_1}$ .................................112
4.9 The influence of the angle of twist on the interaction diagram of natural frequency and buckling moments $\sqrt{m_2}$ .................................113
4.10 The influence of the angle of twist on interaction diagram of natural frequency and shear buckling loads $\sqrt{q_2}$ .................................115
4.11 The influence of the angle of twist on the interaction diagram of natural frequency and buckling moments $\sqrt{m_1}$ .................................116
5.1 A Mindlin plate........................................................................121
5.2 The trapezoidal element coordinate transformation.....................124
5.3a Skew plate subject to uniaxial compression................................127
5.3b Skew plates with SSSS, CCCC and FFFC boundary conditions......127
5.4a A square plate under uniaxial compression with SSSS boundary condition.....................................................................................129
5.4b Dynamic buckling modes of a square plate under uniaxial compression with SSSS boundary condition..............................................130
5.5a A square plate under uniaxial compression with CCCC boundary condition.....................................................................................130
5.5b Dynamic buckling modes of a square plate under uniaxial compression with CCCC boundary condition..............................................131
5.6a A square plate under uniaxial compression with FFFC boundary condition.....................................................................................131
5.6b Dynamic buckling modes of a square plate under uniaxial compression with FFFC boundary condition..............................................................................132
5.7a A skew plate under uniaxial compression with SSSS boundary condition…………………………………………………………………………132
5.7b Dynamic buckling modes of a skew plate under uniaxial compression with SSSS boundary condition…………………………………………………………………………133
5.8 A skew plate under uniaxial compression with CCCC boundary condition……………………………………………………………………………………………………133
5.9 A skew plate under uniaxial compression with FFFC boundary condition……………………………………………………………………………………………………………………………………………………………………134
5.10a A trapezoidal plate subject uniaxial compression…………………..…135
5.10b A trapezoidal plate with SSSS and CCCC boundary conditions……...135
5.11a A trapezoidal plate under uniaxial compression with SSSS boundary condition……………………………………………………………………………………………………………………………………………………………………135
5.11b Dynamic buckling modes of a trapezoidal plate under uniaxial compression with SSSS boundary condition………………………………………………………………136
5.12 A trapezoidal plate under uniaxial compression with CCCC boundary condition………………………………………………………………………………………………………………………………………………………………………………………………………………136
5.13 Polygonal plates subject to isotropic in-plane compression………...138
5.14 A triangular plate under isotropic compression with SSS boundary condition………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………………
5.19 A hexagonal plate under isotropic compression with CCCCCC boundary condition.............................144
5.20 A octagonal plate under isotropic compression with SSSSSSSSS boundary condition.............................144
5.21 A octagonal plate under isotropic compression with CCCCCCCC boundary condition.............................145
5.22a Square plate systems with two rectangular plates......................146
5.22b Square plate systems with two trapezoidal plates......................146
5.23 Natural frequency and buckling loads with respect to the thickness \( t_1 \) for a square plate composed of two rectangular plates or two trapezoidal plates........................................................................................147
5.24 Interaction diagram of a plate system composed of two rectangular elements with FFFC boundary conditions \( (t_1=0.12, t_2=0.1, L_p=1) \).............148
5.25 Interaction diagram of a plate system composed of two trapezoidal elements with FFFC boundary conditions \( (t_1=0.12, t_2=0.1, L_p=1) \).............148
5.26 Dimensions of plate systems composed of four plate elements........149
5.27 Natural frequency and buckling loads with respect to the thickness \( t_1 \) for a square plate composed of four rectangular plates or four trapezoidal plates........................................................................................150
5.28 Plate system composed of four rectangular elements with CCCC boundary conditions \( (t_1=t_3=0.12, t_2=t_4=0.1, L_p=1) \).........................151
5.29 Plate system composed of four trapezoidal elements with CCCC boundary conditions \( (t_1=t_3=0.12, t_2=t_4=0.1, L_p=1) \).........................151
6.1 Geometry of a thin shell panel..................................................157
6.2 Displacements and rotations of shell panels...............................165
6.3 Coordinate systems and dimensions of shell panels.....................170
6.4 The origin element coordinates.................................................171
LIST OF SYMBOLS

The following list comprised those symbols which are used frequently throughout the thesis. Symbols which are confined to only a particular section of the thesis may not be included here, however they are defined where they have been used.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>FC⁰</td>
<td>Fourier C⁰ shape functions</td>
</tr>
<tr>
<td>PC⁰</td>
<td>Polynomial C⁰ shape functions</td>
</tr>
<tr>
<td>FC¹</td>
<td>Fourier C¹ shape functions</td>
</tr>
<tr>
<td>FC¹B</td>
<td>Fourier C¹ shape functions for one-dimensional structures</td>
</tr>
<tr>
<td>PC⁰B</td>
<td>Polynomial C⁰ shape functions for one-dimensional structures</td>
</tr>
<tr>
<td>FC⁰S</td>
<td>Fourier C¹ shape functions for two-dimensional structures</td>
</tr>
<tr>
<td>PC⁰S</td>
<td>Polynomial C⁰ shape functions for two-dimensional structures</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>a free boundary</td>
</tr>
<tr>
<td>C</td>
<td>a clamped boundary</td>
</tr>
<tr>
<td>S</td>
<td>a simply or hard simply supported boundary</td>
</tr>
<tr>
<td>A</td>
<td>cross-sectional area</td>
</tr>
<tr>
<td>D</td>
<td>plate rigidity</td>
</tr>
<tr>
<td>E</td>
<td>Young’s modulus</td>
</tr>
<tr>
<td>G</td>
<td>shear modulus</td>
</tr>
<tr>
<td>I</td>
<td>moment of area</td>
</tr>
<tr>
<td>I₀, J₀</td>
<td>polar moment of area</td>
</tr>
</tbody>
</table>
\( I_1, I_2 \) moment of area in 1 or 2 axis
\( I_x, I_y \) moment of area in \( x \) or \( y \) axis
\( I_\omega \) warping moment of area
\( J \) torsion constant
\( k \) shear correction factor
\( l \) length of beam
\( L \) torque
\( L_a, L_b, L_c \) lengths of plate
\( L_p \) side length of polygonal plate
\( M, N \) moment
\( N_i \) the \( i \)th shape function of element
\( p, p_\xi, p_\eta \) number of additional terms in shape functions over a coordinate
\( P \) axial force
\( q \) distributed forces
\( Q \) shear force
\( r \) rigidity ratio
\( t \) time, thickness
\( T \) kinetic energy
\( u, v, w \) displacement components of structure
\( u_1, u_2, u_3 \) displacement \( u \) of pre-twist beam in 1, 2 and 3 direction
\( v_1, v_2, v_3 \) displacement \( v \) of pre-twist beam in 1, 2 and 3 direction
\( u_1, v_1 \) rotation components of thin shell panel
\( u_b, v_b \) displacements on the bottom end of beam
\( u_t, v_t \) displacements on the top end of a beam
\( U_e \) strain energy due to linear strain
\( U_\sigma \) strain energy due to initial stresses
\( x, y, z \) Cartesian coordinates
\( X_i, Y_i \) \quad \text{values of } x \text{ and } y \text{ coordinates at four corner nodes of a quadrilateral}

\( \alpha \) \quad \text{skew angle of plate}

\( \beta \) \quad \text{inclination angle of trapezoidal plate}

\( \theta \) \quad \text{rotation of beam}

\( \theta_1, \theta_2, \theta_3 \) \quad \text{rotation components in 1, 2 or 3 direction of beam}

\( \theta_{b}, \theta_{t} \) \quad \text{rotation on the bottom or top end of beam}

\( \theta_x, \theta_y \) \quad \text{rotation components of plate}

\( \kappa \) \quad \text{curvature}

\( \lambda \) \quad \text{buckling load intensity factor}

\( \mu \) \quad \text{pre-twisted rate}

\( \nu \) \quad \text{Poisson’s ratio}

\( \xi, \eta \) \quad \text{coordinates in mapped plane}

\( \rho \) \quad \text{density}

\( \sigma_p, \sigma_q \) \quad \text{non-dimensional load parameters of beam}

\( \tau \) \quad \text{tortuosity}

\( \chi \) \quad \text{twist}

\( \omega \) \quad \text{natural frequency}

\( \Omega \) \quad \text{non-dimensional frequency parameter}

\( G \) \quad \text{geometric stiffness matrix}

\( J \) \quad \text{Jacobian matrix}

\( K \) \quad \text{stiffness matrix}

\( M \) \quad \text{mass matrix}

\( N \) \quad \text{matrix of shape functions}

\( \mathbf{q}, \mathbf{r} \) \quad \text{vector of generalized displacements}

\( \mathbf{q}_g \) \quad \text{vector of generalized nodal displacements of beam}
\( q_i \) vector of internal DOFs of beam

\( T, X \) load matrix

\( \delta \) vector of generalized DOFs
CHAPTER 1

INTRODUCTION

1.1 Introduction to the $p$-element Method

The finite element method (FEM) achieves an approximate solution by dividing the domain of interest into a number of smaller sub-domains, called finite elements, and then approximating the solution by employing locally admissible polynomial functions that are piecewise smooth only over each individual sub-domain. The FEM developed by engineers in 1950 was considered in the first three decades as an $h$-version [1-3], which means that the degree $p$ of these polynomials has been arbitrarily set at a low value, typically less than or equal to three, with the mesh size $h$ increased to obtain a better approximation of the exact solution. In the 1970s, the $p$-version of the FEM, which is sometimes referred to the hierarchical FEM, was considered [4-6]. In 1981, the first theoretical paper addressing the $p$-version method was published by Babuska, Szabo and Katz [7], who fixed the mesh size $h$ and increased the degree $p$ of the polynomials to reduce approximation error. In the same year, Babuska and Dorr [8] investigated a combination of both versions and dubbed it “the $hp$-version.” Hence, there are three possible ways of improving the performance of the FEM. The first is to employ the $h$-version to refine the finite element mesh while keeping the degree
of the elements fixed; the second is to employ the \( p \)-version to increase the order of the polynomial shape functions for a fixed mesh; and the third is to employ a combination of the first two methods, that is, the \( h-p \) version, to simultaneously refine the mesh and increase the degree of elements uniformly or selectively.

A serious drawback with ‘standard’ approximation functions is that when element refinement is carried out for the \( h \)-version FEM, completely new shape functions must be generated, and hence all of the calculations have to be repeated. This difficulty can be avoided by considering the shape functions as a series that does not depend on the number of nodes in the mesh. The concept of the \( p \)-version elements is discussed in detail in Refs [9-10]. The advantages of the \( p \)-version elements over their \( h \)-version counterparts are: (i) they have better conditioned matrices [10]; (ii) they do not require a change in the mesh and can easily be used in adaptive analysis [11-12]; (iii) just one element is capable of predicting accurate solutions for a simple structure; (iv) they tend to provide equally accurate results with far fewer degrees of freedom (DOF) [13-18]; and (v) they can overcome certain locking problems [9].

Early \( p \)-version elements mainly adopted shape functions of the Lagrange family, the Jacobi family or the ‘serendipity’ family [9]. More recently, however, Legendre orthogonal polynomials and trigonometric functions have been chosen as the hierarchical and Fourier \( p \)-element shape functions, respectively. Both methods have strengths and weaknesses. When the theory is of \( C^0 \) continuity, especially for a one-dimensional beam element, polynomial series converge much faster than Fourier series. Taking the vibration problems of a curved beam as an example, Zhu [19] indicates that 200 Fourier terms are needed to achieve accurate frequencies, whereas just 12 terms are sufficient for polynomial series. At the
same time, polynomial series usually have a faster convergence rate for certain lower-frequency and buckling modes, whereas Fourier series are more effective in predicting the medium and high modes [10, 20]. However, it is well-known that the order of polynomials cannot be increased indefinitely due to ill-conditioning, and Fourier series appear to be more stable than polynomial series [19]. By considering both the advantages and disadvantages of these two series, the choice of the Fourier series, polynomial series or a combination of the two is carefully made for the shape functions of structures.

1.2 Brief Review of the Application of the $p$-element Method

Considerable progress has been made in the past four decades in the evolution, establishment and advancement of $p$-element methods. Early examples are the papers presented by Surana and his co-authors [21-34], in which the $p$-method was applied to heat condition problems [22, 23, 26, 29] and fluid flow problems [33, 34]. Webb [35-38] used this method to solve electrical-magnetic problems [35]. In addition, there are historical notes on the work of Bardell and his co-authors [39-49], who carried out early research that applied the method to static problems and the vibration problems of structures. Since 1997, Houmat has made considerable progress in applying the $p$-version or $h$-$p$ version method to the investigation of vibration problems in many kinds of structures, including membranes, plates and shells [13-18, 50-58]. Moreover, Ribeiro has investigated the non-linear vibration problems of beams, plates and shells [59-79]. Other available work is that carried out by Han and Petyt [80-81], Beslin [82-83], Woo [84] and Li [85-86]. In 1998, Leung and Chan [87] presented a Fourier $p$-version
element method in their study of the natural frequencies of beams and plates. Leung and Zhu [88-101] later performed a great deal of work on the vibration problems of various structures, including curved beams [91], membranes [88], planes [93], Mindlin plates [94] and three-dimensional thick plates [99-100] using the $p$-element method.

1.3 Scope of Present Study

This thesis follows the work of Leung and Zhu [19, 88-101]. Its main objective is to use the $p$-element method to investigate the buckling and dynamic stability problems of structures based on the vibration results for beams and plates obtained by Leung and Zhu [19, 88-101]. Various kinds of structures, including uniform and non-uniform members, straight and pre-twisted members, one-dimensional beam columns, and two-dimensional plates and shell panels are considered in detail. Either Fourier or polynomial $p$-elements are carefully selected to analyze the different structures, and the natural frequencies, buckling loads and relationship between the frequencies and different types of buckling loads are considered in detail.

The programs given by Zhu [19] have been improved to make them easier and more effective for extensive application. These improved programs can be adopted to solve the problems of straight beams and Mindlin plates using less computational space and time relative to Zhu's earlier programs [19]. They are then extended to solve more complicated structures, such as pre-twisted straight beams and conical and spherical shell panels with higher order polynomial $p$-elements. In contrast to the numerical examples provided by Zhu [19], in the
current study, the boundary conditions for the beams and plates are enriched and include combinations of simply supported (S), clamped (C) and free (F).

The thesis is structured as follows.

Chapter 1: A general introduction to the present study is presented. Brief background on the $p$-element is provided, and the general formulation of the Fourier $p$-element and polynomial $p$-element are introduced.

Chapter 2: This chapter extends the Fourier $C^1$ $p$-element to one-dimensional beam columns. Conservative and follower axial compressive and tensile loads are considered, as are uniform beams and tapered beams with or without end mass.

Chapter 3: Dynamic axial-torsional buckling problems and the flexural-torsional buckling of cantilevered Timoshenko space beams are investigated using the Fourier $C^1$ $p$-element. Two section beams are also analyzed using this method.

Chapter 4: The theory of pre-twisted straight beams is established as a $C^0$ theory. Polynomial $p$-elements are employed as shape functions, and the effects of rigidity and pre-twisted angle on the dynamic stability of pre-twisted straight beams are considered. The effects of axial loads, torque, moments and end shear loads on the vibration and buckling of these beams are investigated in detail.

Chapter 5: An analytical trapezoidal Fourier $p$-element, which was presented by Zhu [19], is introduced to solve the frequencies and buckling loads of Mindlin plates. The triangular and polygonal plates can be divided into combinations of trapezoids and rectangles. The vibration and buckling problems of plate systems with plate elements of different thicknesses are investigated.
Chapter 6: The theory of open shell panels is obtained as a C⁰ theory. The natural frequencies of open cylindrical, conical and spherical shell panels are investigated with polynomial p-elements, and the buckling loads of open cylindrical shell panels subject to axial compressed loads are obtained.

Chapter 7: This chapter concludes the thesis with general closing remarks on the entire research project. Conclusions are drawn about the proposed p-version elements, and directions for further research are proposed.

1.4 General Formulation of p-element

The general formulation of a p-element can be divided into three main steps: 1) shape functions, 2) mapping functions and 3) integration procedures. A comprehensive introduction to the p-version element method can be found in Szabo and Babuska [102].

1.4.1 Shape functions

There are two kinds of finite element shape functions with respect to the inter-element continuity requirements. C⁰ shape functions refer to the minimum requirement for function continuity, and C¹ shape functions to the requirement for function and first-derivative continuity. The three following types of one-dimensional shape functions are used in this thesis.

Type 1:
Fourier-version C⁰ shape functions (FC⁰):
\[ N(\xi) = \left[ 1 - \xi, \xi, \sin(\xi \pi), \sin(2\xi \pi), \ldots \right]. \quad (1.1) \]

Type 2:

Polynomial-version \( C^0 \) shape functions (\( PC^0 \)):

\[ N(\xi) = \left[ \frac{1-\xi}{2}, \frac{1+\xi}{2}, f_m(\xi) \right], \quad (1.2) \]

where the functions \( f_m \) are hierarchical functions that are derived from the Rodrigues form of the Legendre orthogonal polynomials and are given by

\[ f_m(\xi) = \sum_{n=0}^{m/2} \frac{(-1)^n (2m-2n-5)!!}{2^n n!(m-2n-1)!} \xi^{m-2n-1}, \quad m = 3, 4, \ldots, p + 2, \quad (1.3) \]

where \( j!! = j(j-2)\ldots 2 \) or 1 with \( 0!! = (-1)!! = 1 \) (details can be found in Houmat [13]). As \( -1 < \xi < 1 \), by applying transformation \( \xi = 2\xi_0 - 1 \), the shape functions become functions of \( \xi_0 \), and \( 0 < \xi_0 < 1 \). To avoid misunderstanding, \( \xi \) is chosen to represent \( \xi_0 \) in this thesis. The first 14 hierarchical functions are given in Appendix 1.

Type 3:

Fourier-version \( C^1 \) shape functions (\( FC^1 \)):

\[ N(\xi) = \left[ 1 - 3\xi^2 + 2\xi^3, 3\xi^2 - 2\xi^3, \xi (1 - 2\xi + \xi^2) I, \xi (\xi^2 - \xi) I, \xi^2, \sin(\xi \pi), (\xi - \xi^2) \sin(2\xi \pi), \ldots \right]. \quad (1.4) \]
Hence, shape functions $FC^0$, $PC^0$ and $FC^1$ are chosen for direct use for one-dimensional beam-columns, and, as shown in Eq. (1.5), we denote the corresponding one-dimensional shape functions as $FC^0B$, $PC^0B$ and $FC^1B$ respectively:

$$N_i(\xi, \eta) = f_i(\xi),$$

(1.5)

where $i = 1, \cdots, (p_\xi + 2)$.

For two-dimensional plates and shell panels, the shape functions take the form shown in Eq. (1.6), and the corresponding two-dimensional shape functions are defined as $FC^0S$, $PC^0S$ and $FC^1S$, respectively:

$$N_i(\xi, \eta) = f_m(\xi)f_n(\eta),$$

(1.6)

where $m = 1, \cdots, (p_\xi + 2)$, $n = 1, \cdots, (p_\eta + 2)$ and $i = 1, \cdots, (p_\xi + 2)(p_\eta + 2)$.

To avoid misunderstanding, $p_\xi$ is commonly chosen to be equal to $p_\eta$, and $p_\xi$ is written as $p$, in this thesis.

The main body of the thesis comprises Chapters 2-6, with one structural type considered in each. One-dimensional beams are analyzed in Chapters 2-4, two-dimensional plates in Chapter 5, and two-dimensional shell panels in Chapter 6. Table 1.1 shows the DOF and types of shape functions for each structural type. As can be seen in this table, $u$, $v$ and $w$ are the displacement components, $\theta$, $\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$, $\theta_5$, $u_1$ and $v_1$ are the rotations in the corresponding directions, respectively, and
the prime on the displacements denotes the derivative with respect to the neutral axis of the beams. More details on the definitions of the displacements and rotations can be found in each chapter.

<table>
<thead>
<tr>
<th>Structural Type</th>
<th>Chapter 2</th>
<th>Chapter 3</th>
<th>Chapter 4</th>
<th>Chapter 5</th>
<th>Chapter 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Columns</td>
<td>DOF</td>
<td>DOF</td>
<td>DOF</td>
<td>DOF</td>
<td>DOF</td>
</tr>
<tr>
<td></td>
<td>u, u'</td>
<td>u, v, θ,</td>
<td>u, v, w,</td>
<td>w, θ1, θ2, θ3</td>
<td>u, v, w,</td>
</tr>
<tr>
<td></td>
<td></td>
<td>u', v', θ'</td>
<td>θ1, θ2, θ3</td>
<td></td>
<td>θ1, θ2,</td>
</tr>
<tr>
<td>Type of shape functions</td>
<td>FC1B</td>
<td>FC1B</td>
<td>PC0B</td>
<td>FC0S</td>
<td>PC0S</td>
</tr>
</tbody>
</table>

1.4.2 Mapping functions

It is necessary in the one-dimensional case to consider linear mapping alone. Assuming the beam varies from 0 to \( l \), the mapping function and inverse mapping are

\[
\begin{align*}
    x & = \xi l \quad \text{and} \quad \xi = x/l .
\end{align*}
\]  

In the two-dimensional case, if all sides of the mapped elements of the plates or shell panels are straight lines, then linear mapping is generally used. Quadrilateral linear mapping takes the following form.

\[
\begin{align*}
    x & = Q_x (\xi, \eta) = (1-\xi)(1-\eta)X_1 + \xi(1-\eta)X_2 + \xi\eta X_3 + (1-\xi)\eta X_4 , \quad (1.8a) \\
    y & = Q_y (\xi, \eta) = (1-\xi)(1-\eta)Y_1 + \xi(1-\eta)Y_2 + \xi\eta Y_3 + (1-\xi)\eta Y_4 , \quad (1.8b)
\end{align*}
\]
where \((X_i, Y_i)\)  \(i = 1, 2, 3, 4\) are the vertex coordinates of the quadrilateral element.

### 1.4.3 Integration procedures

In computing the stiffness, mass and geometrical stiffness matrices, we have to evaluate integral expressions of the form

\[
I = \iint_{\Omega_\xi} F(x, y) \, dx \, dy = \int_0^1 \int_0^1 F(Q_i(\xi, \eta), Q_j(\xi, \eta)) |J| d\xi d\eta,
\]

where \(\Omega_\xi\) is the original integration domain, and \(|J|\) is the determinant of the Jacobian matrix, which takes the form

\[
J = \begin{bmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \eta} \\
\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \eta}
\end{bmatrix}.
\]

(1.10)
CHAPTER 2

FOURIER $p$-ELEMENTS FOR DYNAMIC STABILITY OF COLUMNS

Using Fourier series as shape functions practically eliminates the ill-conditioning problem associated with higher-order polynomials in structural analyses as confirmed by the newly constructed condition number diagram. This chapter applies the Fourier $C^1$ version $p$-elements to study the dynamic stability of beam-columns. Both conservative and follower forces are considered. The results are compared very well to the exact method of dynamic stiffness. It is found that the Fourier $p$-elements outperform the dynamic stiffness method in terms of versatility in applications and numerical stability at the very low and high ends of the frequency spectrum. New results of follower tension are given. Follower tension buckling under uniformly distributed follower tension is originally reported.
2.1 Introduction

2.1.1 Condition numbers of Fourier $p$-element and polynomial $p$-element

It is well-known that the order of polynomial cannot be increased indefinitely due to ill-conditioning. To improve the ill-conditioning problem, Leung and Chan [87] used the $C^0$ Fourier enriched shape functions $f_1(\xi)=1-\xi$, $f_2(\xi)=\xi$, $f_i(\xi)=\sin(i-2)\pi\xi$, $(i\geq 3)$ to predict the axial vibration frequencies of a beam. The sine functions represent the internal DOF. On the other hand, the $C^0$ Legendre orthogonal polynomials are widely used in formulation of $p$-version hierarchical elements [40-41]. The Rodrigues formula for the Legendre polynomials in $\xi\in(-1,1)$ is given by $f_m = \frac{1}{2^m m!} \frac{d^m}{d\xi^m}(\xi^2-1)^m$. The condition number $CN$ defined in Cook et al [103] is used to measure the ill-conditioning in a coefficient matrix. The smaller $CN$, the more stable is the matrix. The mass matrices of a uniform beam vibrating axially for a Fourier $p$-element and a Legendre $p$-element are respectively,

$$
\begin{bmatrix}
\frac{1}{3} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{j\pi} \\
\frac{1}{3} & \frac{(-1)^{j+1}}{j\pi}
\end{bmatrix}, \quad \text{sym}
\begin{bmatrix}
\frac{1}{3} & \frac{1}{6} & -\frac{1}{6} & \frac{1}{30} & 0 & 0 \\
\frac{1}{3} & -\frac{1}{6} & -\frac{1}{30} & 0 & 0 \\
-\frac{2}{5x3x5} & 0 & -\frac{1}{9x7x5} & 0 \\
\frac{2}{7x5x3} & 0 & \frac{1}{11x9x7} & \ddots & \ddots & \ddots 
\end{bmatrix}
$$

where $\left[ \begin{array}{c} \vdots \\ \delta_{ij}/2 \\
\vdots \end{array} \right]$ denotes a row vector and $\delta_{ij}$ is the Kronecker delta which is one when $i=j$ and zero otherwise. The condition numbers of the Fourier and the Legendre mass matrices denoted by ‘+’ and ‘•’ respectively are given in Figure
2.1 for various matrix size $N$. It is evident that the Fourier $p$-mass matrix is much more stable than that of the Legendre $p$-element. Indeed, the condition number grows like $N^{1.4}$ for the Fourier $p$-element and $N^{4.2}$ for the Legendre $p$-element in this particular case.

![Figure 2.1 Condition numbers](image)

**Figure 2.1 Condition numbers**

### 2.1.2 The concept of follower forces

A follower force is an applied force whose direction changes according to the deformed shape during the course of deformation. Early studies on follower force were made by Ziegler [104] and Beck [105], who considered a cantilevered beam subject to a tangential follower force at the free end. Stability problems of columns under uniformly distributed follower loading was presented by Leipholz [106]. An extensive study has been made by Bolotin [107] on non-conservative
problems of elastic stability of structures. In 1967, Herrmann [108] wrote a general review on follower forces problems. Later, the vibration and stability of a non-uniform Timoshenko beam were investigated by Irie et al using a transfer matrix method [109]. Meanwhile, a distributed follower force stability analysis was given by Chen and Ku [110] using finite element method. Lee [111] studied the dynamic stability of a tapered cantilevered beam on an elastic foundation. Leung et al [112] investigated the piecewise non-uniform Timoshenko column using a dynamic stiffness method. Moreover, non-conservative stability problems of multi-step non-uniform columns were investigated by Li [113], in which both concentrated follower forces and variably distributed follower forces were considered. Recent development on stability of beams under follower compression can be found in the work of Wang [114], Diondorov [115], Li [116], Marzani [117], Goyal [118] and Kim [119]. The readers are referred to Langthjem et al [120], Maurizi et al [121] and Elishakoff [122] for a comprehensive review of follower compression of beam-columns. Follower tension stability problems of beams were early studied by the author and Leung [123-124] using Fourier $p$-elements method. Later, follower tension buckling of beam-columns was considered by Leung [125] using exact spectral dynamic stiffness method.

The Fourier $p$-elements have been widely used [15-18, 87-93]. It has been successfully applied to various structures of beams and plates of different shapes and many three-dimensional problems. We shall use Fourier $p$-elements to study the dynamic stability of beam columns and structures under conservative and follower forces in this chapter. The formulation of the Fourier $p$-elements for beam columns is briefly discussed in Section 2.2 and numerical examples are given in Section 2.3. Interaction diagrams are compared with the exact dynamic stiffness method [126]. New results of follower tension are given and follower
tension buckling under uniformly distributed follower tension is originally reported. Following the results reported by the author and Leung [123-124] using p-element, effects of follower tension have been considered in Leung’s work using power series [125].

2.2 Governing Equation of Beam Columns

Consider a one-dimensional bending beam element vibrating with frequency $\omega$ subject to a constant second order axial force $P$ as shown in Figure 2.2.

![Figure 2.2 A plane bending beam subject to axial force $P$](image)

$q_g$, denoted the generalized nodal displacements of the beam.

$$q_g = [u_l, u'_l, u_r, u'_r]^T,$$  \hspace{1cm} (2.1)

where $u_l$ and $u_r$ are the displacements in $y$ direction on the left ($x=0$) and right ($x=l$) end of the beam, and the prime in $(u'_l, u'_r)$ means the gradient of the deformation in axis $y$ with respect to the neutral axis $x$.

The energy of the one-dimensional beam-columns in the presence of axial load $P$
is

\[
U = \frac{1}{2} \int_0^l \begin{bmatrix} u & u' & u'' \end{bmatrix}^T \begin{bmatrix} -\omega^2 \rho A & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & EI \end{bmatrix} \begin{bmatrix} u \\ u' \\ u'' \end{bmatrix} \, dx ,
\]

(2.2)

where \( E, I, \rho, A \) and \( l \) are Young’s modulus, area moment of inertia, density, cross-sectional area and the length of the beam, respectively.

### 2.3 Fourier \( p \)-elements for Beam Columns

#### 2.3.1 Shape functions

Fourier-version \( \mathcal{C}^1 \) shape functions satisfying the displacement and slope continuity at element interface can be used to analyze the vibration of a two-node beam [87]. The shape functions will be extended to solve the dynamic buckling problems of a beam.

\[
\mathbf{N} = \mathbf{N}_u = \begin{bmatrix} f_1 & g_1 & f_2 & g_2 & b_j \end{bmatrix},
\]

(2.3)

where \( f_1, f_2, g_1, g_2 \) and \( b_j \) are given as follows:
\[ f_i = 1 - 3\xi^2 + 2\xi^3, \]
\[ g_1 = \xi \left( 1 - 2\xi + \xi^2 \right), \]
\[ f_2 = 3\xi^2 - 2\xi^3, \]
\[ g_2 = \xi \left( \xi^2 - \xi \right), \]
\[ b_j = \left( \xi - \xi^2 \right) \sin j\pi \xi, \ j = 1, 2, \ldots, (p) \]
\[ \xi = \frac{x}{l}, (0 < x < l) \]

where \(0 < \xi < 1\) is the non-dimensional length, \(p\) is the number of the internal degree of freedom of the beam.

Therefore, the lateral displacement \(u\) of a beam subject to constant axial force \(P\) can be given by

\[ u = Nq, \quad (2.5) \]

where \(N\) is the shape functions for the space beams, \(q = \begin{bmatrix} q_x \\ q_t \end{bmatrix}\), \(q_x\) is the generalized nodal displacements defined in Eq. (2.1) and the vector \(q_t\) with the size of \(p\) is the internal degree of freedoms of the beams,

\[ q_t = [t_u]_{x,p}, \quad (2.6) \]

2.3.2 Stiffness, mass and geometric stiffness matrices

The equation of the eigenvalue problem is given by
\[
\left( K - \omega^2 M - PG \right) q = 0,
\]

where \( P \) is positive when in compression and is negative when under tension loads, \( q \) is the vector of nodal displacements and the stiffness matrix \( K \), mass matrix \( M \) and geometric matrix \( G \) are given by

\[
K = \sum_e K^e, \quad M = \sum_e M^e, \quad G = \sum_e G^e,
\]

\[
K^e = E I \int_0^l \left( \frac{d^2 N}{dx^2} \right)^T \left( \frac{d^2 N}{dx^2} \right) dx,
\]

\[
M^e = \rho A l \int_0^l N^T N dx,
\]

\[
G^e = \int_0^l \left( \frac{dN}{dx} \right)^T \left( \frac{dN}{dx} \right) dx.
\]

The entries of the stiffness, mass and geometric stiffness matrices are given in Appendix 2 (Eq. A2.1-A2.3).

### 2.3.3 Formulation of follower force

An externally applied force which changes the direction of application (but not the magnitude) during deformation is called a follower force. A follower force can be generated by the thrust of a solid-propellant rocket motor [120] or by a water jet issued from a nozzle box [122]. Figure 2.3 shows the concept of follower forces, in which \( q \) is the distributed forces and \( P \) is the concentrated forces.
2.3.3.1 Concentrated follower forces

Consider a beam subject to a concentrated follower force. The lateral displacement $v(x, t)$ of a beam subject to constant axial force $P$ is given by

$$v(x, t) = Nq e^{i\omega t}, \quad (2.12)$$

where $q$ is the generalized nodal displacements. In the conventional finite element method, when an element is subjected to concentrated loads, the nodal force vector of the element is given by

$$P^e = \sum_j N^T (\xi_j) P_j. \quad (2.13)$$

When the axial force at the top node ($x=l$) is a follower force, then, the nodal force vector $P^e$ is modified to
\[
\overline{P}^e_{\text{tot}} = -EI \frac{\partial^3 v}{\partial x^3} = P^e_{\text{tot}} - P^e \frac{\partial v}{\partial x}.
\] (2.14)

Therefore, the nodal force vector has to be modified as

\[
P^e = -N^T \left( \xi_j \right) \left|_{j=3} \right. P^e \left|_{j=3} \right. \frac{\partial N}{\partial x} \left|_{x=l} \right. \cdot q^e_{\text{tot}}.
\] (2.15)

By substituting the shape functions into Eq. (2.15), the follower element force \(P^e\) can be reduced to the modification of the geometric matrix, i.e. the third row, the fourth column of the geometric matrix should include the term \(G^e(3,4) = -l\).

When the follower force acted on the bottom node \((x=0)\), the term is \(G^e(1,2) = l\).

2.3.3.2 Uniformly distributed follower forces

Consider a beam subject to distributed follower forces, the lateral displacement \(v(x, t)\) of a beam subject to constant axial force \(P\) is given by Eq. (2.12). The geometric matrix for the element is,

\[
G^e = \int_0^l H(x) \left( \frac{dN}{dx} \right)^T \left( \frac{dN}{dx} \right) dx,
\] (2.16)

where \(H(x) = l - x\), for a element subjected to uniformly distributed loads. When an element is subjected to distributed loads, the nodal force vector of the element is given by
When the distributed force is a follower force, then, the nodal force vector $\mathbf{P}^e$ is modified to

$$
\mathbf{P}^e_{e^{tot}} = -EI \frac{\partial^3 \mathbf{v}}{\partial x^3} = \mathbf{P}^e_{e^{tot}} - \mathbf{P}^e \frac{\partial \mathbf{v}}{\partial x}. \quad (2.18)
$$

Therefore, the nodal force vector has to be modified

$$
\mathbf{P}^e = -q_o \int_0^1 \mathbf{N}^T(\xi) \left( \frac{\partial \mathbf{N}}{\partial \xi} \right) d\xi. \quad (2.19)
$$

One can find that the follower element force $\mathbf{P}^e$ can be reduced to the modification of the geometric matrix, i.e. the following matrix should be subtracted from the geometric matrix: $\mathbf{G}^e = \int_0^1 \mathbf{N}^T(\xi) \left( \frac{\partial \mathbf{N}}{\partial \xi} \right) d\xi$. The entries of the element $\mathbf{G}^e$ and $\mathbf{G}^e$ for uniformly distributed forces are shown in Appendix 2 (Eq. A2.4-A2.5).

### 2.4 Numerical Examples

Numerical examples of buckling problems of beams subjected to conservative or follower forces are considered in this study. Dynamic stiffness and substructure method has been used by Leung [126] to solve the same problems. The results calculated by Fourier $p$-element will be compared with Leung’s results. The
relationships of $\Omega = \left( \frac{\rho A l^4 \omega^2}{EI} \right)^{1/4}$, $\sigma_p = \left( \frac{Pl^2}{EI} \right)^{1/2}$ and $\sigma_q = \left( \frac{ql^3}{EI} \right)^{1/2}$ are shown in the figures below, in which the dot-dash lines are results of the dynamic stiffness method and the dash lines are given by the Fourier $p$-element. Frequency $\Omega$, load $\sigma_p$ and $\sigma_q$ are non-dimensional parameters in this chapter.

### 2.4.1 Buckling problems of beams under conservative concentrated loads

a) *Compression*

Consider a beam-column subject to conservative axial compression as the first example. The interaction diagram for a cantilever column is compared with the exact dynamic stiffness matrix method in Figure 2.4a. The narrow full lines are the results from the exact dynamic stiffness method and the thick dashed lines the Fourier $p$-element. Only one element is used in both cases. No difference can practically be found. Figures 2.4b-d show the interaction diagrams produced by the Fourier $p$-elements for various boundary conditions.

![Figure 2.4a Clamped-Free beam subject to conservative axial compression](image-url)

Figure 2.4a Clamped-Free beam subject to conservative axial compression
Chapter 2: Dynamic stability of columns

Figure 2.4b Hinged-Hinged beam subject to conservative axial compression

Figure 2.4c Clamped- Hinged beam subject to conservative axial compression
b) **Tension**

Consider a beam column subject to conservation axial tension. The cantilever case agrees well with the dynamic stiffness method in Figure 2.5a. Fourier $p$-elements are used to produce the interaction diagrams for the other boundary conditions in Figure 2.5b-d. It is obvious that the natural frequencies increase with the tension force monotonically but the center of curvature changes side giving a point of inflexion for each curve.
Figure 2.5a Clamped-Free beam subject to conservative axial tension

Figure 2.5b Hinged-Hinged beam subject to conservative axial tension
2.4.2 High modes comparison for buckling problems of beams under follower forces

Consider a cantilever beam under follower forces. The interaction diagrams are compared with the exact dynamics stiffness method in Figures 2.6. Flutter loads
are clearly observed. It is found that the Fourier $p$-elements are numerically stable even for very high modes while the exact dynamic stiffness method is numerically unstable due to the cancellation of large numbers in both the numerator and the denominator of the frequency functions in the non-symmetric dynamic stiffness resulting in some blurred areas in the diagrams.

Figure 2.6a High modes of Clamped-Free beam subject to concentrated follower compression
2.4.3 Buckling problems of columns under follower forces

a) Concentrated follower tension

Consider a cantilever beam under follower tension. It is interesting to note that the natural frequency decreases with increasing follower tension in the beginning of the first mode and has not been reported before. The exact dynamic stiffness method gives the same result after putting the follower compression as negative. An enlarged region for the uniform column subject concentrated follower tension is given in Figure 2.7 where negative frequency axis is also shown for clarity. It is clear that no buckling occurs.
b) *Uniformly distributed follower forces*

Consider a cantilever beam under uniformly distributed follower forces. To verify the correctness of the formulae given in Appendix 2 (Eq. A2.4-A2.5), we cross check the first and second modes under uniformly distributed follower compression by comparing with Leipholz [106] who gave $\omega_0=12.36$, $(\omega_1)^2=485.52$, and $g_{fl}=40.05$ as shown in Figure 2.8. The results by using Fourier $p$-element are $\omega_0=12.366$, $(\omega_1)^2=485.35$ and $g_{fl}=40.05$ which agree well with results of Leipholz.
Chapter 2: Dynamic stability of columns

Figure 2.8 Eigenvalue curve $g = g(\omega^2)$ of a column subject to uniformly distributed follower compression by Fourier $p$-element

2.4.4 Interaction diagrams of beams under concentrated conservative tension and follower tension

a) Concentrated tension

From Figure 2.9a-d it is found that there is not any difference for hinged-hinged column, clamped-free column and clamped-clamped column when subject to concentrated follower tension or conservative tension but discrepancy exists for clamped-free condition. The phenomenon that the first buckling load decreases initially against the frequency only happens to the clamped-free column subject to follower tension.
Figure 2.9a Clamped-Free column subject to concentrated follower tension

Figure 2.9b Hinged-Hinged column subject to concentrated follower tension
Chapter 2: Dynamic stability of columns

Figure 2.9c Clamped-Hinged column subject to concentrated follower tension

Figure 2.9d Clamped-Clamped column subject to concentrated follower tension

b) Distributed tension

It is shown from Figure 2.10a-d that for clamped free column there is much difference between the results of columns when subject to follower force and conservative force. There is much less difference between follower force and conservative force for other boundary conditions when compared with
clamped-free column. The phenomenon that the first buckling load decreases against the frequency only occurred to clamped-free column subject to follower distributed tension. The regions near buckling at the first mode are enlarged in Figures 2.11. It is observed that the buckling follower tension is at \( g = 21.13 \). Follower tension buckling can be generated using an example of a vessel loaded by a liquid shown in Fig 2.12.

![Figure 2.10a Clamped-Free column subject to distributed follower tension](image)

![Figure 2.11 Enlarged portions at the first mode under uniformly distributed follower tension](image)
Figure 2.10b Hinged-Hinged column subject to distributed follower tension

Figure 2.10c Clamped-Hinged column subject to distributed follower tension
2.4.5 Interaction diagrams of tapered beams

Consider a tapered beam-column of the dimensions as shown in Figure 2.13,

\[ A = A_0 \left(1 + \frac{1}{10} x \right), \quad I = I_0 \left(1 + \frac{1}{10} x \right)^3. \quad (2.20) \]
The Fourier $p$-element matrices are given by

$$K^e = EI \int_0^l \left[ 1 + \frac{1}{10} x \right] \left( \frac{d^2 N}{dx^2} \right)^T \left( \frac{d^2 N}{dx^2} \right) dx, \tag{2.21}$$

$$M^e = \rho A_0 \int_0^l \left[ 1 + \frac{1}{10} x \right] N^T N dx, \tag{2.22}$$

$$G^e = \int_0^l \left( \frac{dN}{dx} \right)^T \left( \frac{dN}{dx} \right) dx. \tag{2.23}$$

After integration, the entries of stiffness and mass matrices are obtained as Eq.(A2.6-A2.7) given in Appendix 2.

The uniform beam and tapered beam shown in Figure 2.13 are taken as comparison examples. Both beams are subject to conservative and follower axial
loads and the interaction diagrams are compared in Figures 2.14a-c with solid lines for uniform beam and dash lines for tapered beam. Due to the small 10% taper, the buckling and flutter loads change only a little in the lower modes but the changes become significant for higher modes. Consider this example again but reverse the direction of the follower force so that the beams are suffered from follower tension. The results are presented in Figure 2.14d-f.

Figure 2.14a Uniform beam and tapered beam under conservative axial compression
Figure 2.14b Uniform beam and tapered beam under follower concentrated compression

Figure 2.14c Uniform column and tapered column subject to distributed follower compression
Figure 2.14d Uniform beam and tapered beam under conservative axial tension

Figure 2.14e Uniform and tapered columns under concentrated follower tension
Chapter 2: Dynamic stability of columns

2.4.6 Interaction diagrams of columns with end mass

Consider the influence of a concentrated mass $M$ as shown in Figure 2.15, such that $\frac{MEI}{\rho Al^4}=0$ or 1. When the concentrated mass is attached on the free end of the column, the modification of the mass matrix is to include the term $M$ at the entry $(3, 3)$ of the mass matrix. When the columns are subject to follower concentrated or distributed tensions, the interaction diagrams are compared in Figures 2.16 with solid lines for $\frac{MEI}{\rho Al^4}=0$ and dash lines for $\frac{MEI}{\rho Al^4}=1$. It can be found that the end mass has significant influence on the buckling modes of column subject to concentrated force or distributed force.
Figure 2.15 Clamped-Free column subject to follower tension with end mass

Figure 2.16a Column subject to concentrated follower tension with or without end mass
Figure 2.16b Column subjected to distributed follower tension with or without end mass

Figure 2.16c Column subject to concentrated follower compression with or without end mass
2.4.7 Interaction diagrams of columns subject to combined concentrated follower tension and distributed follower tension

Consider the problem of clamped-free columns subject to a concentrated follower tension \( P \) and a uniformly distributed follower tension \( q \). The equation of the eigenvalue problem is given by

\[
\left( K - \omega^2 M + PG_p + qG_q \right) d = 0. \quad (2.24)
\]

The interaction diagrams for the buckling of the columns subject to combined follower tension are shown in Figure 2.17a and 2.18a. Only the first mode is shown in this study. In Figure 2.17b and 2.18b, the colored lines are contour plot on plane \( \sigma_P = 0 \) and plane \( \sigma_q = 0 \), respectively; the thick dashed lines are the mode 1 interaction diagrams for the buckling of the columns subject to distributed...
follower tension or concentrated follower tension only. It can be seen from Figure 2.17b and Figure 2.18b that the isolines tend close to the dashed lines and the accuracy of the 3D plot of the relationship of $\lambda$, $\sigma_p$ and $\sigma_q$ is proved.

Figure 2.17a 3D plot of $\left(\Omega, \sigma_q, \sigma_p\right)$ for column subject to combined follower tension

Figure 2.17b Contour plot for $\left(\Omega, \sigma_q, \sigma_p\right)$
Chapter 2: Dynamic stability of columns

Figure 2.18a 3D plot of \((\Omega, \sigma_p, \sigma_q)\) for column subject to combined follower tension

Figure 2.18b Contour plot for \((\Omega, \sigma_p, \sigma_q)\)
2.5 Conclusion

The Fourier $C^1 p$-elements have been extended to the dynamic stability analysis of beam-columns under conservative and non-conservative concentrated and distributed axial forces. The problem of ill-conditioning associated with polynomial $p$-elements is eliminated and the numerical stability associated with the dynamic stiffness method for very high frequency and axial load is avoided. New results of follower tension are given and follower tension buckling under uniformly distributed follower tension is originally reported.
CHAPTER 3

FOURIER $p$-ELEMENTS FOR DYNAMIC STABILITY OF SPACE BEAMS

Using Fourier series as shape functions in a finite element analysis practically eliminates the ill-conditioning problem associated with higher-order polynomials in structural analysis as confirmed by the newly constructed condition number diagram. This chapter extends the Fourier $C^1$ sine-version $p$-elements to study the axial-torsional buckling and flexural-torsional buckling against the frequency of vibration of space beams. It is found that the natural frequency and buckling loads of space beams obtained by Fourier $p$-elements agreed well with the analytical solutions. Non-uniform sections can also be analyzed by this method. The interaction diagrams due to vibration frequency, axial force, buckling torque and moments are considered in this chapter. For the axial-torsional buckling of space beams, mode pairs about major and minor principal axes form kinks in the frequency-torque diagram and even cross each other in the compression-torque diagram. A two-section column is investigated numerically. Finally, flexural-torsional buckling of cantilevered I-beam due to end moment, end lateral force and distributed lateral forces are studied.
3.1 Introduction

Leung [127-128] indicated that when a space beam is subjected to an end torque, and the bending rigidities of the beam in the major and minor axis are different, the initial shear stresses will not be equally shared in major and minor axis as Ziegler [129] and Kim et al [130] proposed, but having the share ratio $I_x: I_y$. Basing on the unequally sharing torque theory, Leung considered the buckling problems of space beams subject to combined bi-axial moments and torque [131] or subject to axial loads and torsional loads [132] using dynamic stiffness method. Besides, the axial-moment buckling [133] or non-conservative buckling of space frames [134] is considered by Leung using a power series dynamic stiffness method.

The Fourier $C^1_p$-element has been used to analyze dynamic stability problems of plane beams in Chapter 2. We shall extend this method to study the axial-torsional buckling and flexural-torsional buckling of cantilevered space beams in this Chapter adopting Leung’s unequally torque sharing theory. The description of the governing equation of space beams is discussed in Section 3.2, in which external buckling loads including axial force, torque and moments are considered. The formulation of the Fourier $p$-elements for beam-columns is briefly discussed in Section 3.3. Numerical examples of axial-torsional buckling of doubly symmetric thin beams are considered in Section 3.4. The interaction influence between buckling torque and axial loads are discussed in detail. Then, a two-section beam is investigated. Finally, numerical results of flexural-torsional buckling of a cantilevered I-beam are obtained. The end moment, end shear force and uniformly distributed shear force on the buckling of I-beam are considered.
3.2 Description of the Problem and Governing Equation of Space Beams

Consider a space beam element vibrating with frequency $\omega$ subject to a constant axial force $P$, torque $L$ and end moment $M$ and $N$. We follow the Euler-Bernoulli assumption with warping considered. A beam element with second order axial force and torque and end moment is shown below:

The generalized nodal displacement vector $\mathbf{q}_g$ of the space beam has the form

$$\mathbf{q}_g = [u_r, v_r, \theta_r, u_r', v_r', \theta_r', u_l, v_l, \theta_l, u_l', v_l', \theta_l']^T,$$

(3.1)

where $(u_r, v_r)$ and $\theta_r$ are the displacements and rotation on the right side of the space beam ($z=0$), and the prime on $(u_r', v_r', \theta_r')$ means the gradient of the deformation in plane $(x, y)$ with respect to the coordinate $z$, and $(u_l, v_l, \theta_l, u_l', v_l', \theta_l')$ denotes the values on the left side of the beam ($z=l$).
In the presence of axial force $P$, and inertia force $\omega^2 \rho A$ due to harmonic vibration of frequency, the total potential energy due to initial torque $L$, end moment $M$ and $N$ is given by Leung [131-132],

$$U = \frac{1}{2} \int_0^z \begin{bmatrix} u \\ v \\ \theta \\ u' \\ v' \\ \theta' \\ u^* \\ v^* \\ \theta^* \end{bmatrix} \begin{bmatrix} -\omega^2 \rho A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\omega^2 \rho A & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\omega^2 \rho J_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & P & 0 & M & 0 & -\frac{I_L}{J_0} & 0 \\
0 & 0 & 0 & 0 & N & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & M & N & GJ & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{I_L}{J_0} & 0 & E I_y & 0 & 0 \\
0 & 0 & 0 & -\frac{I_L}{J_0} & 0 & 0 & 0 & E I_y & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E I_o \end{bmatrix} \begin{bmatrix} u \\ v \\ \theta \\ u' \\ v' \\ \theta' \\ u^* \\ v^* \\ \theta^* \end{bmatrix} \, dz,$$

(3.2)

where $z$ is the neutral axis, a prime denotes differentiation with respect to $z$, $u$ and $v$ are vibration amplitudes of displacements along the $x$ and $y$ axis of the element $l$, $\theta$ is the angle of twist as shown in Figure 3.1, and $E$, $G$, $\rho$ and $A$ are Young’s modulus, shear modulus, mass density and cross-sectional area respectively. And $I_x = \int y^2 \, dA$ and $I_y = \int x^2 \, dA$ are the principal moments of cross-sectional area $A$. $I_w$ is the warping moment of area. $J_0 = I_x + I_y$ is the polar moment of area which is an approximation to the torsion constant $J$ for regular sections.
3.3 Formulation of Fourier $p$-elements for Space Beams

3.3.1 Shape functions

Fourier-version $C^1$ shape functions can be used to analyze the buckling problems of a two-node plane beam [123-124]. The shape functions will be extended to solve the dynamic buckling problems of a space beam in this chapter. The lateral displacement $u$ of a beam subject to constant axial force $P$, torque $L$ and moment $M$ and $N$ is given by

$$u = Nq,$$  \hspace{1cm} (3.3)

where $N$ is the shape functions for the space beams, $q = \begin{bmatrix} q_g \\ q_i \end{bmatrix}$, $q_g$ is the generalized nodal displacements defined in Eq. (3.1) and $q_i$ is the internal freedoms of the beams.

$$q_i = [t_u \quad t_v \quad t_\theta]^T,$$  \hspace{1cm} (3.4)

where $t_u$, $t_v$ and $t_\theta$ corresponding to the internal degrees of freedom with respect to $u$, $v$ and $\theta$.

The Fourier sine-version shape functions satisfying the displacement and slope continuity at element interface for spaced beams of solid cross-sections are as following:
\[
\mathbf{N}(\xi) = \begin{bmatrix}
\mathbf{N}_u \\
\mathbf{N}_v \\
\mathbf{N}_0
\end{bmatrix} = \begin{bmatrix}
f_1 & 0 & 0 & g_1 & 0 & 0 & f_2 & 0 & 0 & g_2 & 0 & 0 & b_j & 0 & 0 \\
0 & f_1 & 0 & 0 & g_1 & 0 & 0 & f_2 & 0 & 0 & g_2 & 0 & 0 & b_j & 0 \\
0 & 0 & f_1 & 0 & 0 & g_1 & 0 & 0 & f_2 & 0 & 0 & g_2 & 0 & 0 & b_j
\end{bmatrix},
\]

(3.5)

where

\[
f_i = 1 - 3\xi^2 + 2\xi^3, \quad f_2 = 3\xi^2 - 2\xi^3, \quad g_i = \xi(1 - 2\xi + \xi^2)l, \quad g_2 = \xi(\xi^2 - \xi)l,
\]

\[
b_j = (\xi - \xi^2)\sin j\pi \xi, \quad j = 1, 2, 3, \ldots, p, \quad \xi = \frac{z}{l} \quad (0 < z < l),
\]

and \(0 < \xi < 1\) is the non-dimensional length, \(p\) is the number of internal degrees of freedom, and \(l\) is the length of the beam.

### 3.3.2 Stiffness, mass and geometric matrices

The equation of the eigenvalue problem of the space beams is given by

\[
(K - \omega^2 M - PG_P - LG_L - MG_M - NG_N)q = 0,
\]

(3.6)

where \(P\) is positive when in compression, \(q\) is the generalized nodal displacements. For non-trivial solution, \(q \neq 0\) and

\[
det \left( K - \omega^2 M - PG_P - LG_L - MG_M - NG_N \right) = 0
\]

gives a relation between

\[
\left( \omega^2, P, L, M, N \right)
\]

when buckling occurs. Then the stiffness matrix \(K\), mass matrix \(M\) and geometric matrices \(G_P, G_L, G_M\) and \(G_N\) can be obtained from Eq. (3.2) as follows,
\[ K = \sum_{\epsilon} K^{\epsilon}, \quad M = \sum_{\epsilon} M^{\epsilon}, \quad G = \sum_{\epsilon} G^{\epsilon}, \quad \] (3.7)

where

\[ K^{\epsilon} = \int_{0}^{L} \left[ \begin{array}{cccc} GJ & 0 & 0 & 0 \\ 0 & EI_{y} & 0 & 0 \\ 0 & 0 & EI_{z} & 0 \\ 0 & 0 & 0 & EI_{w} \end{array} \right] \begin{bmatrix} N_{\theta} \\ N_{u} \\ N_{v} \\ N_{\phi} \end{bmatrix} \, dz 
= \frac{1}{l} \int_{0}^{L} \left[ \begin{array}{cccc} GJl^{2} & 0 & 0 & 0 \\ 0 & EI_{y} & 0 & 0 \\ 0 & 0 & EI_{z} & 0 \\ 0 & 0 & 0 & EI_{w} \end{array} \right] \begin{bmatrix} N_{\theta}(\xi) \\ N_{u}(\xi) \\ N_{v}(\xi) \\ N_{\phi}(\xi) \end{bmatrix} \, d\xi, \]

(3.8)

\[ M^{\epsilon} = \int_{0}^{L} \rho \left[ \begin{array}{cccc} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & J \end{array} \right] \begin{bmatrix} N_{u} \\ N_{v} \\ N_{\phi} \end{bmatrix} \, dz 
= \frac{1}{l} \int_{0}^{L} \rho \left[ \begin{array}{cccc} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & J \end{array} \right] \begin{bmatrix} N_{u}(\xi) \\ N_{v}(\xi) \\ N_{\phi}(\xi) \end{bmatrix} \, d\xi, \]

(3.9)

\[ G^{\rho} = P \int_{0}^{L} \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] \begin{bmatrix} N_{\theta}' \\ N_{v}' \end{bmatrix} \, dz 
= \frac{P}{l} \int_{0}^{L} \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] \begin{bmatrix} N_{\theta}'(\xi) \\ N_{v}'(\xi) \end{bmatrix} \, d\xi, \]

(3.10)
The entries of the stiffness matrix \( K' \), mass matrix \( M' \) and geometric matrices \( G'_{p}, G'_{L}, G'_{M}, G'_{N} \) are given in Appendix 3.
3.4 Numerical Examples of Axial Torsional Buckling of Space Beams

Numerical examples of buckling problems of cantilever beams subjected to axial force and buckling torque as shown in Figure 3.2 are considered in this study.

![Figure 3.2 A cantilevered beam subject to axial loads and torque](image)

For simplicity in our study we use $\Omega^{1/4}$, $P^{1/2}$ and $L$ to represent the non-dimensional parameters $\left(\frac{\rho Al^4 \omega^2}{EI_x}\right)^{1/4}$, $\left(\frac{P l^2}{EI_x}\right)^{1/2}$ and $\left(\frac{L l}{E J_0}\right)$ instead. The relationships of $\Omega^{1/4}$, $P^{1/2}$ and $L$ will be computed in the figures below.

3.4.1 Buckling problems of uniform beams

Consider a uniform cantilever beam with rigidity $I_x/I_y=0.5$ as the first example. The material and geometric properties are $E=G=A=l=\rho=1$, $I_w=0$ and $I_y=1$.

3.4.1.1 Eigen values compared with analytical solutions

To prove the accuracy of Fourier $p$-element, firstly the buckling loads and natural
frequencies for the cantilever beam subject to axial compression are calculated and compared with the analytical solutions in this section. Every pair of buckling modes in the major and minor principal axes ($i$ and $i+1$, where $i$ is the number of the mode, $i=1,3,5,...$) satisfies the relation $(P_{cr}^{i/2})^{i+1} = \left(\frac{I_y}{I_x}\right)^{1/2} \left(P_{cr}^{1/2}\right)^i$. Likewise, every pair of natural frequencies has the relation $(\Omega_{cr}^{i/4})^{i+1} = \left(\frac{I_y}{I_x}\right)^{1/4} \left(\Omega_{cr}^{1/4}\right)^i$. The present results of buckling loads and natural frequencies are compared with the analytical results of Timoshenko [135] and Timoshenko [136] in Table 3.1 and Table 3.2. It can be seen that the results calculated by Fourier $p$-element fit well with the analytical solutions. In all numerical examples, we use 20 Fourier terms.

Table 3.1 Buckling axial compressive load parameters $P_{cr}^{i/2}$ comparison

<table>
<thead>
<tr>
<th>Mode $i$ (in major axis)</th>
<th>Present $p=20$</th>
<th>Analytical [135]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.1107</td>
<td>1.1107</td>
</tr>
<tr>
<td>3</td>
<td>3.3322</td>
<td>3.3322</td>
</tr>
<tr>
<td>5</td>
<td>5.5538</td>
<td>5.5536</td>
</tr>
<tr>
<td>7</td>
<td>7.7756</td>
<td>7.7750</td>
</tr>
<tr>
<td>9</td>
<td>9.9978</td>
<td>9.9965</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mode $i+1$ (in minor axis)</th>
<th>Present $p=20$</th>
<th>Analytical [135]</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.5708</td>
<td>1.5708</td>
</tr>
<tr>
<td>4</td>
<td>4.7124</td>
<td>4.7124</td>
</tr>
<tr>
<td>6</td>
<td>7.8543</td>
<td>7.8540</td>
</tr>
<tr>
<td>8</td>
<td>10.9964</td>
<td>10.9956</td>
</tr>
<tr>
<td>10</td>
<td>14.1391</td>
<td>14.1372</td>
</tr>
</tbody>
</table>

Table 3.2 Natural frequency parameters $\Omega_{cr}^{i/4}$ comparison

<table>
<thead>
<tr>
<th>Mode $i$ (in major axis)</th>
<th>Present $p=20$</th>
<th>Analytical [136]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.5768</td>
<td>1.5768</td>
</tr>
<tr>
<td>3</td>
<td>3.9473</td>
<td>3.9473</td>
</tr>
<tr>
<td>5</td>
<td>6.6051</td>
<td>6.6051</td>
</tr>
<tr>
<td>7</td>
<td>9.2465</td>
<td>9.2462</td>
</tr>
<tr>
<td>9</td>
<td>11.8888</td>
<td>11.8879</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mode $i+1$ (in minor axis)</th>
<th>Present $p=20$</th>
<th>Analytical [136]</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.8751</td>
<td>1.8751</td>
</tr>
<tr>
<td>4</td>
<td>4.6941</td>
<td>4.6941</td>
</tr>
<tr>
<td>6</td>
<td>7.8549</td>
<td>7.8548</td>
</tr>
<tr>
<td>8</td>
<td>10.9960</td>
<td>10.9956</td>
</tr>
<tr>
<td>10</td>
<td>14.1383</td>
<td>14.1372</td>
</tr>
</tbody>
</table>
3.4.1.2 Interaction diagram between buckling loads and natural frequency

Consider the interaction diagram between the buckling load and natural frequency. Since the rigidity ratio is not equal in major and minor axis, some low modes in the minor axis intersect with high modes in the major axis as seen from Figure 3.3a (i.e. mode 6 intersect with mode 7, mode 8 with mode 9). When the beam is subject to axial tension, Figure 3.3b shows that the first modes about major and minor axes are very close to each other for tension-frequency curves. The distance between different mode pairs becomes larger for increasing compression. It is shown in Figure 3.3c for frequency-torque buckling, except mode 1, every mode pairs contain some kinks, and the number of the kinks appeared is equal to \((i-1)/2\), \(i \ (i=3,5,7\ldots)\) is the odd number of the buckling mode. The phenomena associate with frequency-torque buckling only and not frequency-compression buckling. The kinks signal the possible energy exchange.

![Interaction diagram between buckling compression and natural frequency](image-url)

Figure 3.3a Interaction diagram between buckling compression and natural frequency
3.3.1.3 The influence on frequency-load diagram taking the other load as a parameter

Consider the interaction diagram between one load and natural frequency while taking the other load as a parameter. The first and second buckling modes are studied in this section. The influence between the buckling torque and axial
compression is considered first. It is shown in Figure 3.4a and Figure 3.4b, when the load parameter is increased, the mode pair gets apart from each other, and at the same time the two modes are dropping. The first pair of pure axial compression buckling loads in Figure 5b is \( P^{1/2} = 1.1107 \) and \( P^{1/2} = 1.5708 \). It is shown in Figure 3.4a that the first pair of pure torsional buckling loads is \( L = 1.1776 \) and \( L = 1.7843 \). Figure 3.4a shows when the buckling torque \( L \) is changed from 1.1 to 1.2, which covers the value of \( L = 1.1776 \), the first branch of the frequency-compression diagram is disappeared. And when \( L \) is over 1.7843, the second branch will also be disappeared. Figure 3.4b shows when \( P^{1/2} > 1.1107 \), the first branch disappeared, and when \( P^{1/2} > 1.5708 \), the second branch will be disappeared, too.

![Figure 3.4a The influence of the buckling torque on axial compression](image-url)
Figure 3.4c shows the frequency-tension diagram when taking the buckling torque as a parameter. It is shown in Figure 3.4c that when $L=1.1$ changes to $1.2$, which covers the first buckling torque $L=1.1776$, the first qualitative change appeared. A second qualitative change appeared between $L=1.7$ and $L=1.8$, a range covers the second buckling torque $L=1.7843$. Figure 3.4d shows the influence of the axial tension on the interaction diagram between frequency and buckling torque. When the axial load is in tension, the two modes get close to each other in contrast with compression. Both the natural frequencies and buckling torque are increased by the axial tension.
Figure 3.4c The influence of the buckling torque on axial tension

Figure 3.4d The influence of the axial tension on buckling torque
3.4.1.4 The interaction diagram between buckling torque and axial loads

The interaction diagram between buckling torque and axial loads is considered in this section. The kinks appeared in Figure 3.5a are similar to Figure 3.3c. Mode pairs can cross over for some values of the applied torque.

Figure 3.5a The interaction diagram between the buckling torque and compression

Figure 3.5b The interaction diagram between the buckling torque and tension
3.4.1.5 Three-dimensional interaction diagram

The three-dimension interaction diagram of $\Omega^{1/4}$, $P^{1/2}$ and $L$ are given in Figure 3.6a and 3.6b.

Figure 3.6a 3D interaction diagram for axial compression

Figure 3.6b 3D interaction diagram for axial tension
3.4.2 Buckling problems of two section step beam

Consider a two section step cantilever beam as shown in Figure 3.7. The thicker and the thinner section of the step beam have the rigidities of \(4EI_y\), \(4EI_x\) and \(EI_y\), \(EI_x\) respectively. And the rigidity ratio is \(I_x/I_y=2\). The length of every section is \(l=2\).

In this section \(\Omega^{1/4}\), \(P^{1/2}\) and \(L\) are used to represent the non-dimensional parameters \(\left(\frac{\rho Al^4 \omega^2}{EI_{y\text{thick}}}\right)^{1/4}\), \(\left(\frac{Pl^2}{EI_{y\text{thick}}}\right)^{1/2}\) and \(\frac{Ll}{EI_{y\text{thick}}}\) instead, which the superscript ‘thick’ means the properties of the thicker section for the step beam. Figure 3.8a and Figure 3.8b give the interaction diagrams between the buckling loads and natural frequency. The influence of the buckling torque on axial compression is shown in Figure 3.9. Three-dimensional frequency-compression-torque interaction diagram is shown in Figure 3.10. The similar phenomena can be found as the uniform beam studied in Section 3.4.1.

![Figure 3.7 A cantilevered two-section step beam](image-url)
Figure 3.8a Interaction diagram between axial compression and natural frequency of the two section beam

Figure 3.8b Interaction diagram between axial tension and natural frequency of the two section beam
Figure 3.8c Interaction diagram between buckling torque and natural frequency of the two section beam

Figure 3.9 The influence of buckling torque on axial compression for the two section beam
3.5 Numerical Examples of Flexural-Torsional Buckling of Space Beams

Thin-walled structural members may fail in a flexural-torsional buckling mode, in which the member suddenly deflects laterally and twists out of the plane of loading. This form of buckling may occur in a member that has low lateral bending and torsional stiffnesses compared with its stiffness in the plane of loading. A comprehensive review on this kind of buckling can be found in Trahair’s book [137]. This section is concerned with the elastic buckling of cantilevers under end moments, end shear loads, or uniformly distributed shear loads. The loading cases are shown in Figure 3.11 in detail.
Chapter 3: Dynamic stability of space beams

For simplicity in our study we use $\Omega^{1/4}$, $M^{1/2}$, $N^{1/2}$, $Q^{1/2}$ and $q^{1/2}$ to represent the non-dimensional parameters $\left(\frac{\rho A l^4 \omega^2}{E I_y}\right)^{1/4}$, $\left(\frac{M l}{E I_y}\right)^{1/2}$, $\left(\frac{N l}{E I_y}\right)^{1/2}$, $\left(\frac{Q l^2}{E I_y}\right)^{1/2}$ and $\left(\frac{q l^3}{E I_y}\right)^{1/2}$ instead. The relationships of $\Omega^{1/4}$ with $M^{1/2}$, $N^{1/2}$, $Q^{1/2}$ or $q^{1/2}$ will be given in the figures below.

Consider a thin-walled I-section beam with the dimensions as shown in Figure 3.12.

The beam length $l=1\text{m}$. The cross-sectional area $A=10.3\text{cm}^2$, the principal moments of area are $I_y=132.858\text{cm}^4$ and $I_x=5.591\text{cm}^4$ and the warping moment of
area is \( I_w = 119.513 \text{cm}^6 \). Young’s modulus \( E = 2.1 \times 10^7 \text{N/cm}^2 \), shear modulus \( G = 8.4 \times 10^6 \text{N/cm}^2 \), torsional constant \( J = 1.416 \text{cm}^4 \) and the polar moment of area \( J_0 = I_x + I_y = 138.449 \text{cm}^4 \).

To prove the accuracy of Fourier \( p \)-element, firstly the non-dimensional buckling load parameters \( M^{1/2} \) and \( N^{1/2} \) for cantilevered beam with end moment \( M \) or \( N \) are solved and compared with analytical solutions shown in Trahair [137]. The analytical buckling loads have the form:

\[
M_{cr} = \left( \frac{2n - 1}{2} \right) \pi \sqrt{EI_wGJ \left( 1 + \left( \frac{2n - 1}{4l^2} \right)^2 \right)}, \quad (3.14)
\]

\[
N_{cr} = \left( \frac{2n - 1}{2} \right) \pi \sqrt{EI_wGJ \left( 1 + \left( \frac{2n - 1}{4l^2} \right)^2 \right)}, \quad (3.15)
\]

where \( n \) is the mode number of the buckling loads.

The present non-dimensional buckling load parameters \( M^{1/2} \) and \( N^{1/2} \) are compared with the analytical solutions [137] in Table 3.3. 20 Fourier terms are used in this section. It is found the results of Fourier \( p \)-elements agree well with the analytical results. The dynamic stability problems of an I-section beam subject to end moment \( M \), end load \( Q \) and uniformly distributed loads \( q \) are considered in this section. The moments due to end load \( Q \) or distributed loads \( q \) have the forms \( M_Q = Q(l-z) \) or \( M_q = q(l-z)^2/2 \), where \( 0 < z < l \) is the beam length varied from the clamped end to free end along the cantilevers. The first five mode values of frequency parameters \( \Omega^{1/4}_w \), \( \Omega^{1/4}_N \), and buckling load parameters \( Q^{1/2} \) and \( q^{1/2} \) of the I-section beam are listed in Table 3.4. The interaction diagram between natural
frequency $\Omega^{1/4}$ and buckling moments $M^{1/2}$ is given in Figure 3.13. As shown in Figure 3.13, the initial moment $M^{1/2}$ hardens the torsion modes and soften the flexural modes of the beam simultaneously, which has been found by Leung [138] using a dynamic stiffness method. The monotony of the slope variation of the characteristic curves is lost due to the interaction. Similar phenomenon can be found in the interaction diagram of frequency and forces $N^{1/2}$, $Q^{1/2}$ or $q^{1/2}$ given in Figure 3.14-3.16, respectively.

Figure 3.17 shows the first interaction mode between the end moment $M^{1/2}$ and end moment $N^{1/2}$, shear loads $Q^{1/2}$ or $q^{1/2}$. It is found when $M^{1/2}$ is very small, the value of $N^{1/2}$, $Q^{1/2}$ or $q^{1/2}$ are almost equal to the buckling loads for the beam only subject one type of them. While when $M^{1/2}$ is close to the buckling value $M^{1/2}=0.3243$, the value of $N^{1/2}$, $Q^{1/2}$ or $q^{1/2}$ is decreased suddenly to 0. By comparing the slope values of the three curves $M^{1/2}$-$N^{1/2}$, $M^{1/2}$-$Q^{1/2}$ and $M^{1/2}$-$q^{1/2}$, minimum effect of moment $M$ on moment $N$ can be found, then followed by end load $Q$ and distributed load $q$.

Table 3.3 Buckling load parameters $M^{1/2}$ and $N^{1/2}$ comparison

<table>
<thead>
<tr>
<th>Mode number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M^{1/2}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Present $p=20$</td>
<td>0.3243</td>
<td>0.6106</td>
<td>0.8821</td>
<td>1.1632</td>
<td>1.4522</td>
<td>1.7462</td>
</tr>
<tr>
<td>Analytical [137]</td>
<td>0.3243</td>
<td>0.6106</td>
<td>0.8820</td>
<td>1.1631</td>
<td>1.4520</td>
<td>1.7459</td>
</tr>
<tr>
<td>$N^{1/2}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Present $p=20$</td>
<td>0.1469</td>
<td>0.2766</td>
<td>0.3995</td>
<td>0.5268</td>
<td>0.6577</td>
<td>0.7909</td>
</tr>
<tr>
<td>Analytical [137]</td>
<td>0.1469</td>
<td>0.2766</td>
<td>0.3995</td>
<td>0.5268</td>
<td>0.6577</td>
<td>0.7908</td>
</tr>
</tbody>
</table>
Table 3.4 Frequency and buckling load parameters $\Omega_M^{1/2}$, $\Omega_N^{1/2}$, $Q^{1/2}$ and $q^{1/2}$ of the I-section beam

<table>
<thead>
<tr>
<th>Mode number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega_M^{1/4}$ Present $p=20$</td>
<td>1.8218</td>
<td>1.8751</td>
<td>3.3364</td>
<td>4.6941</td>
<td>4.7147</td>
<td>6.1476</td>
</tr>
<tr>
<td>$\Omega_N^{1/4}$ Present $p=20$</td>
<td>0.8493</td>
<td>1.8218</td>
<td>2.1261</td>
<td>3.3364</td>
<td>3.5577</td>
<td>4.7147</td>
</tr>
<tr>
<td>$Q^{1/2}$ Present $p=20$</td>
<td>0.5351</td>
<td>0.9607</td>
<td>1.3779</td>
<td>1.8040</td>
<td>2.2381</td>
<td>2.6783</td>
</tr>
<tr>
<td>$q^{1/2}$ Present $p=20$</td>
<td>1.0060</td>
<td>1.7871</td>
<td>2.5848</td>
<td>2.7757</td>
<td>2.9609</td>
<td>3.6070</td>
</tr>
</tbody>
</table>

Figure 3.13 Interaction diagram between natural frequency $\Omega^{1/4}$ and end moment $M^{1/2}$
Figure 3.14 Interaction diagram between natural frequency $\Omega^{1/4}$ and end moment $N^{1/2}$

Figure 3.15 Interaction diagram between natural frequency $\Omega^{1/4}$ and end load $Q^{1/2}$
Figure 3.16 Interaction diagram between natural frequency $\Omega^{1/4}$ and distributed load $q^{1/2}$

Figure 3.17 The relation between end moment $M^{1/2}$ and end moment $N^{1/2}$, end load $Q^{1/2}$ or distributed load $q^{1/2}$
3.6 Conclusion

The Fourier $C^1$ $p$-elements have been extended to the dynamic axial-torsional buckling and flexural-torsional buckling of space beams. The buckling loads and natural frequencies are agreed well with analytical solutions. The interaction influence between buckling torque and axial loads of a rectangular section beam are studied. Moreover, the interaction diagram between the buckling moments, end lateral load and distributed lateral loads of an I-section beam are investigated.
CHAPTER 4

POLYNOMIAL $p$-ELEMENTS FOR DYNAMIC STABILITY OF PRE-TWISTED STRAIGHT BEAMS

Free vibration and buckling of pre-twisted beams exhibit interesting coupling phenomena between compression, moments and torque and have been the subject of extensive research due to their importance as models of wind turbines and helicopter rotor blades. The chapter investigates the influence of all kinds of initial stresses due to compression, shears, moments and torque on the natural vibration of pre-twisted straight beam based on the Timoshenko theory. The derivation begins with the three-dimensional Green strain tensor. The nonlinear part of the strain tensor is expressed as a product of displacement gradient to derive the strain energy due to initial stresses. The Frenet formulae in differential geometry are employed to treat the pre-twist. The strain energy due to elasticity and the linear kinetic energy are obtained in classical sense. From the variational principle, the governing equations and the associated natural boundary conditions are derived. The Legendre orthogonal polynomials are chosen to solve the vibration and buckling problems of the pre-twisted straight beam in this chapter. It is noted that the first mode increases together with the pre-twisted angle but the second
decreases seeming to close the first two modes together for natural frequencies and compressions. The gaps close monotonically as the angle of twist increases for natural frequencies and buckling compressions. However, unlike natural frequencies and compressions, the closeness is not monotonic for buckling shears, moments and torques.

4.1 Introduction

Free vibration and buckling of pre-twisted beams [139-166] exhibit interesting coupling phenomena between compression, moments and torque and have been the subject of extensive research due to their importance as models of wind turbines and helicopter rotor blades. Rosard [139], Troesch et al [140] and Diprima and Handelman [141] were about the first few papers that investigated the natural vibration of pre-twisted beams. Subsequent applications to blade and coupling vibration were investigated by Carnegie [142-143], Slyper [144], Anliker and Troesch [145], Dawson [146] and Lin [147]. Gupta and Rao [148], Sisto and Chang [149], Yardimoglu and Yildirim [150] developed the finite elements to study vibration problems of pre-twisted beams or bladings. Celep and Turham [151] included the influence of shear and rotator inertia on the vibration of pre-twisting beams. Rosen et al [152-153] used principal coordinates and Onipede et al [154] and Balhaddad and Onipede [155] studied the three dimensional behaviors. Petrov and Geradin [156] presented a nonlinear theory. Banerjee [157-158] introduced an exact solution method of dynamic stiffness. Many works are existed to study vibration problems of pre-twisted beams under axial loadings, including Chen and Keer [159], Lee [160], Liao and Huang [161] and Sakar and Sabuncu [162]. Some recent developments on buckling problems of pre-twisted
beams can be found in [163-166]. However, fewer references are contributed to
dynamic stability problems of pre-twisted beams subject to shears, moments and
torques.

We aim at formulating the influence of all kinds of initial stresses due to
compression, shears, moments and torque on the natural vibration of pre-twisted
straight beam based on the Timoshenko theory. The formulation can be extended
to system of many pre-twisted beams from the equilibrium and compatibility
consideration. We begin with the three-dimensional Green strain tensor. The
nonlinear part of the strain tensor is expressed as a product of displacement
gradient to derive the strain energy due to initial stresses. The Frenet formulae in
differential geometry are employed to treat the pre-twist. The strain energy due to
elasticity and the kinetic energy are obtained in classical sense. From the
variational principle, the governing equations and the associated natural boundary
conditions are derived.

The $p$-element method is a kind of method to improve the performance of finite
element method by increasing the order of the polynomial shape functions for a
fixed mesh. The $p$-element method has been successfully applied to vibration
problems of beams, plates and shells [13-58]. The element displacement is
described by linear shape functions plus a variable number of hierarchical
functions which are forms of orthogonal Legendre polynomials or trigonometric
series. The linear shape functions are used to define the two nodal displacement of
the beam and the hierarchical functions are used to provide additional degrees of
freedom to the interior of the beam element. The results can be improved by
increasing the terms of the additional functions. Only one element is needed to
achieve excellent results. When the beam is of $C^0$ continuity, the element using
trigonometric functions as the shape functions cannot overcome the shear locking problems, and the Legendre polynomials are proved to be convergent much faster than Fourier series [19], the Legendre orthogonal polynomials [13] are chosen to solve the vibration and buckling problems of the pre-twisted straight beam in this chapter. The vibration problems of Rosen [152] and Banerjee [157-158] are taken as the first example. The natural frequencies obtained by the $p$-element agree well with Rosen [152] and Banerjee’s results [158]. Secondly, to prove the accuracy of the present theory, the results of frequency and buckling shears and compressions of a pre-twisted rectangular cross-section beam are compared with results of ANSYS ($h$-version), and good agreement is found. The influence of the pre-twist angle and rigidity ratio on the first two natural frequencies, buckling loads of the beam with rectangular cross-sections are considered. It is interesting to note that the first natural frequency and compression modes increases together with the angle of twist but the second decreases seeming to close the first two modes together for natural frequencies and compressions. The gaps reduce monotonically as the angle of twist increases for natural frequencies and buckling compressions. However, unlike natural frequencies and compressions, the closeness is not monotonic for buckling shears, moments and torques.

4.2 Incremental Strain Analysis

In general three-dimensional elasticity analysis, the Green strain tensor is given by

$$e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_k} \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right),$$

(4.1)
which can be written in engineering form in linear and nonlinear parts as

$$\mathbf{e} = \mathbf{e}^0 + \mathbf{e}^1,$$  \hfill (4.2)

in which

$$\begin{bmatrix}
    e_{11}^0 \\
    e_{22}^0 \\
    e_{33}^0 \\
    2e_{12}^0 \\
    2e_{23}^0 \\
    2e_{31}^0
\end{bmatrix}, \quad \begin{bmatrix}
    e_{11}^1 \\
    e_{22}^1 \\
    e_{33}^1 \\
    2e_{12}^1 \\
    2e_{23}^1 \\
    2e_{31}^1
\end{bmatrix}, \quad \text{for } e_j^0 = \frac{1}{2} \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right), e_j^1 = \frac{1}{2} \frac{\partial u_k}{\partial x_j} \frac{\partial u_j}{\partial x_i}.  \hfill (4.3)$$

However, the nonlinear part \( \mathbf{e}^1 \) can be rewritten as

$$\mathbf{e}^1 = \frac{1}{2} \mathbf{H} \mathbf{\phi} = \frac{1}{2} \begin{bmatrix}
    0 & 0 & 0 \\
    0 & \frac{\partial u_i}{\partial x_i} & 0 \\
    0 & 0 & \frac{\partial u_i}{\partial x_3} \\
    \frac{\partial u_i}{\partial x_1} & \frac{\partial u_i}{\partial x_2} & 0 \\
    0 & \frac{\partial u_i}{\partial x_2} & \frac{\partial u_i}{\partial x_3}
\end{bmatrix} \begin{bmatrix}
    \frac{\partial u_i}{\partial x_1} \\
    \frac{\partial u_i}{\partial x_2} \\
    \frac{\partial u_i}{\partial x_3}
\end{bmatrix}.  \hfill (4.4)$$

where \( \mathbf{u}(x,y,z) = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \) and \( \mathbf{\phi} = \begin{bmatrix} \frac{\partial \mathbf{u}}{\partial x_1} \\ \frac{\partial \mathbf{u}}{\partial x_2} \\ \frac{\partial \mathbf{u}}{\partial x_3} \end{bmatrix} \).

It is easily proved that

$$\left( \mathbf{dH} \right) \mathbf{\phi} = \mathbf{H} \left( \mathbf{d\phi} \right).  \hfill (4.5)$$
Therefore,
\[
\text{d}e^i = \frac{1}{2} (\text{d}H) \phi + \frac{1}{2} H (\text{d}\phi) = (\text{d}H) \phi - H (\text{d}\phi). \tag{4.6}
\]

Let the stress vector be
\[
\sigma = \{\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31}\}^T, \tag{4.7}
\]

then, by direct expansion,
\[
\begin{align*}
\begin{bmatrix}
\frac{\partial u}{\partial x_1} & 0 & 0 & \frac{\partial u}{\partial x_2} & 0 & \frac{\partial u}{\partial x_3}
\end{bmatrix}
& \begin{bmatrix}
\sigma_{11}
\sigma_{22}
\sigma_{33}
\sigma_{12}
\sigma_{23}
\sigma_{31}
\end{bmatrix}^T

&= dH^T \sigma = \begin{bmatrix}
\sigma_{11} \frac{\partial u}{\partial x_1} + \sigma_{12} \frac{\partial u}{\partial x_2} + \sigma_{13} \frac{\partial u}{\partial x_3}
\sigma_{21} \frac{\partial u}{\partial x_1} + \sigma_{22} \frac{\partial u}{\partial x_2} + \sigma_{23} \frac{\partial u}{\partial x_3}
\sigma_{31} \frac{\partial u}{\partial x_1} + \sigma_{32} \frac{\partial u}{\partial x_2} + \sigma_{33} \frac{\partial u}{\partial x_3}
\end{bmatrix}

&= \begin{bmatrix}
\sigma_{11} \text{I} & \sigma_{12} \text{I} & \sigma_{13} \text{I}
\sigma_{21} \text{I} & \sigma_{22} \text{I} & \sigma_{23} \text{I}
\sigma_{31} \text{I} & \sigma_{32} \text{I} & \sigma_{33} \text{I}
\end{bmatrix}

&= \begin{bmatrix}
\frac{\partial u}{\partial x_1}
\frac{\partial u}{\partial x_2}
\frac{\partial u}{\partial x_3}
\end{bmatrix}

&= [\sigma] \text{d}\phi. \tag{4.8}
\end{align*}
\]

When the initial stress vector \( \sigma \) in Eq. (4.7) is sufficiently considered, a second order analysis will be completed. In the following analysis, the non-vanishing strains and stresses are \( \{e_{31}, e_{32}, e_{33}\} \) and \( \{\sigma_{31}, \sigma_{32}, \sigma_{33}\} \) respectively to satisfy the Timoshenko assumption.
Chapter 4: Dynamic stability of pre-twisted straight beams

4.3 Geometry

4.3.1 The geometry of a general curve with a pre-twisted rate

For a general curve in space whose Cartesian coordinate vector \( \mathbf{v} = \mathbf{v}(z) \) is determined by the arc-length coordinate \( z \) (in the \( \mathbf{e}_3 \) direction to be defined later), then, from differential geometry, the generalized curvatures \( p_c, q_c \) and \( r_c \) are defined by

\[
\frac{\partial \mathbf{v}}{\partial z} = \mathbf{v}' + K \mathbf{v}, \quad K = \begin{bmatrix} 0 & r_c & -q_c \\ -r_c & 0 & p_c \\ q_c & -p_c & 0 \end{bmatrix}, \quad \begin{bmatrix} p_c \\ q_c \\ r_c \end{bmatrix} = \begin{bmatrix} \kappa \sin \chi \\ \kappa \cos \chi \\ \tau + \mu \end{bmatrix}, \quad \mu = \chi', \quad (4.9)
\]

where \( \kappa \) and \( \tau \) and the curvature and tortuosity of the centerline and \( \chi \) is the twist of the cross-section to bring the principal axes in alignment with the principal normal and bi-normal, and the prime on \( \chi \) denotes the derivation with respect to \( z \).

If the beam is uniformly twisted when unstressed, then \( \chi = \mu z \) and the Frenet formulae is

\[
\begin{bmatrix} p_c \\ q_c \\ r_c \end{bmatrix} = \begin{bmatrix} \kappa \sin \mu z \\ \kappa \cos \mu z \\ \tau + \mu \end{bmatrix}, \quad (4.10)
\]

Consider a twisted straight beam when \( \kappa = \tau = 0 \) but \( \mu \neq 0 \) whose cross-section is shown in Figure 4.1. The principal axes are along \( x \left( \mathbf{e}_1, \mathbf{N} \right) \) and \( y \left( \mathbf{e}_2, \mathbf{B} \right) \) respectively and \( \mathbf{N} \) and \( \mathbf{B} \) are the unit vectors of normal and bi-normal. Along the centerline, the position \( \tilde{r}(z) \), normal \( \mathbf{N}(z) \), bi-normal \( \mathbf{B}(z) \) and tangent \( \mathbf{T} \).
vectors are given respectively by

\[
\vec{r}(z) = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}, \quad \vec{N}(z) = \begin{bmatrix} \cos \mu z \\ -\sin \mu z \\ 0 \end{bmatrix}, \quad \vec{B}(z) = \begin{bmatrix} \cos \mu z \\ 0 \\ 0 \end{bmatrix}, \quad \vec{T} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \quad (4.11)
\]

The position vector \( \vec{r}(x,y,z) \) or \( \vec{r}(R,\theta,z) \) of a point on the plane \( z = \text{constant} \) at polar coordinates \( (R,\theta) \) away from the centerline is given by

\[
\vec{r} = \vec{r} + R(-\cos \theta \vec{N} + \sin \theta \vec{B}) = \begin{bmatrix} -R \cos \mu z \cos \theta + R \sin \mu z \sin \theta \\ R \sin \mu z \cos \theta + R \cos \mu z \sin \theta \\ z \end{bmatrix}. \quad (4.12)
\]

In this chapter, the coordinate system \( (x,y,z) \) may be interchangeably used with \( (x_1, x_2, x_3) \). In all cases, the Jacobian is equal to one.

Consider a straight beam of length \( l=2\pi \) as shown in Figure 4.1(a). The rate of pre-twist \( \mu \) of unit in rad/m is assumed to be constant along the length. Then, the rotation angle measured from the bottom side to the top side of the beam can be defined as the pre-twisted angle \( \vec{\mu} = \mu l \). The straight beam with pre-twisted ratio \( \mu = 0.125 \) and 0.25 are shown in Figure 4.1(b) and 4.1(c), having the corresponding pre-twisted angle \( \vec{\mu} = \pi/4 \) and \( \pi/2 \), respectively. Figure 4.1(d) gives a perspective view of Figure 4.1(c) and Figure 4.1(e) depicts the local coordinate system.
Figure 4.1 A pre-twisted straight beam: (a) straight beam, (b) twist rate $\mu=0.125$, (c) twist rate $\mu=0.25$, (d) perspective view of Figure 4.1(c), (e) local coordinate system.
When the centerline is straight, $\tau = \kappa = 0$, and the Frenet formulae becomes

$$\frac{d}{dz} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} = \begin{bmatrix} 0 & \mu & 0 \\ -\mu & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} \text{ or } \mathbf{\ddot{e}}' = \mathbf{K} \mathbf{\ddot{e}}$$

where $\mathbf{K} = \begin{bmatrix} 0 & \mu & 0 \\ -\mu & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

4.3.2 Displacement, strain and external force

The coordinate system attached to the positive face across the centerline along the principle axes for a straight prismatic beam is shown in Figure 4.2, assuming the Timoshenko’s plane-remain plane cross-section during deformation. Axis 3 is along the centerline, axes 1 and 2 are the principal axes making 1-2-3 a right hand triad.

The displacements and rotations of a rectangular cross-section along the centerline at $z$ are denoted by $\mathbf{r}(z)$ where
\[ r(z) = \begin{bmatrix} v \\ 0 \end{bmatrix}, \]  

where \( v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \) and \( \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \).

The Timoshenko assumptions for plane-remain-plane during deformation give

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & -y \\
0 & x & 0
\end{bmatrix}
\begin{bmatrix}
u_1(z) \\
v_2(z) \\
v_3(z)
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & -x \\
0 & x & 0 \\
y & -x & 0
\end{bmatrix}
\begin{bmatrix}
\theta_1(z) \\
\theta_2(z) \\
\theta_3(z)
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & -y \\
0 & 1 & 0 & 0 & 0 & x \\
0 & 0 & 1 & y & -x & 0
\end{bmatrix}
r(z). \tag{4.15}
\]

The Frenet formulae for total differential with respect to \( z \) give

\[
\begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix}
= u' + Ku
= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & -y \\
0 & 1 & 0 & 0 & 0 & x \\
0 & 0 & 1 & y & -x & 0
\end{bmatrix}
\begin{bmatrix}
u_1(z) \\
u_2(z) \\
u_3(z)
\end{bmatrix}
+ \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & x & y & 0
\end{bmatrix}
r. \tag{4.16}
\]

A prime denotes local differentiation with respect to \( z \). Also,

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 0
\end{bmatrix}
r(z) \quad \text{and} \quad \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
r(z). \tag{4.17}
\]

The non-vanishing strain components \( e_{3i} \) are
\[ e = \begin{bmatrix} e_{31} \\ e_{32} \\ e_{33} \end{bmatrix} = \begin{bmatrix} u_{3,1} \\ u_{3,2} \end{bmatrix} + \frac{\partial}{\partial z} \begin{bmatrix} u_{1} \\ u_{2} \\ u_{3} \end{bmatrix} \]

\[
= \begin{bmatrix} 0 & \mu & 0 & 0 & -1 & 0 & 0 & 0 & 0 & y \\ -\mu & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & x \\ 0 & 0 & 0 & \mu x & \mu y & 0 & 0 & 0 & 1 & y & -x & 0 \end{bmatrix} \begin{bmatrix} r \\ r' \end{bmatrix} = b(x,y) \begin{bmatrix} r \\ r' \end{bmatrix},
\]

where a comma subscript represents partial derivative and a prime denotes differential with respect to \( x_3 \). For elastic modulus matrix
\[
Y = \begin{bmatrix} G & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & E \end{bmatrix},
\]

\( G, E \) are the usual elastic modulus, the non-vanishing stresses \( \sigma_{3i} \) are given by

\[
\sigma = \begin{bmatrix} \sigma_{31} \\ \sigma_{32} \\ \sigma_{33} \end{bmatrix} = Ye,
\]

which induce the resistance forces \( Q_i, i = 1, 2, 3 \) on the positive face as shown in Figure 4.2 as

\[
\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \int_{\alpha \text{dx} \text{dy}} \begin{bmatrix} 0 & \mu GA & 0 & 0 & -GA & 0 & GA & 0 & 0 \\ -\mu GA & 0 & 0 & GA & 0 & 0 & 0 & GA & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & EA \end{bmatrix} \begin{bmatrix} v \\ v' \end{bmatrix},
\]

as well as the resistance moments \( M_i, i = 1, 2, 3 \),
where $A$ is the cross-sectional area, $I_i = \int y^2 \, dA = \int x_i^2 \, dA$, $I_2 = \int x_i^2 \, dA$, and $I_0 = I_1 + I_2$, and $J$ is the torsion constant. Other area integrals are assumed to be zeros due to double symmetry. Without loss of generality, the effective shear area factor $k$ is taken as one for simplicity, because one can always simply change $GA$ to $kGA$ when needed as will be done in the numerical examples later.

The initial stresses are given by

$$\sigma^0 = \begin{bmatrix} \sigma_{31}^0 \\ \sigma_{32}^0 \\ \sigma_{33}^0 \end{bmatrix} = \begin{bmatrix} Q_1 / A \\ Q_2 / A \\ Q_3 / A \end{bmatrix} \begin{bmatrix} 0 & 0 & -y \\ 0 & 0 & x \\ y & -x & 0 \end{bmatrix} \begin{bmatrix} M_1 / I_1 \\ M_2 / I_2 \\ M_3 / I_0 \end{bmatrix}. \quad (4.22)$$

Putting the displacement gradient in matrix form, on has

$$\begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} = -\int R \sigma \, dx \, dy = \begin{bmatrix} EI_1 & 0 & 0 \\ 0 & EI_2 & 0 \\ 0 & 0 & GI_0 \end{bmatrix} \{\theta^i\}$$

$$= \begin{bmatrix} 0 & EI_1 \mu & 0 & 0 & 0 & EI_1 & 0 & 0 \\ -EI_2 \mu & 0 & 0 & 0 & 0 & 0 & EI_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & GJ \end{bmatrix} \begin{bmatrix} \theta \\ v' \end{bmatrix}, \quad (4.21)$$
4.4 Energy

The steady state kinetic energy with the vibration frequency $\omega$ is given by

$$T = \frac{\omega^2}{2} \int \rho r^7 D_m r dr,$$  \quad (4.24)  

where

$$D_m = \begin{bmatrix}
A & 0 & 0 & 0 & 0 & 0 \\
0 & A & 0 & 0 & 0 & 0 \\
0 & 0 & A & 0 & 0 & 0 \\
0 & 0 & 0 & I_1 & 0 & 0 \\
0 & 0 & 0 & 0 & I_2 & 0 \\
0 & 0 & 0 & 0 & 0 & I_0
\end{bmatrix}. \quad (4.25)$$

The elastic strain energy density per unit length due to the non-vanishing initial stresses is given by Eq. (4.7)
Chapter 4: Dynamic stability of pre-twisted straight beams

\[ U_{\sigma}(x) = \frac{1}{2} \int_A \begin{bmatrix} u_{r1} \\ u_{r2} \\ u_{r3} \end{bmatrix}^T \begin{bmatrix} 0 & 0 & \sigma_{31}^0 I \\ 0 & 0 & \sigma_{32}^0 I \\ \sigma_{31}^0 I & \sigma_{32}^0 I & \sigma_{33}^0 I \end{bmatrix} \begin{bmatrix} u_{r1} \\ u_{r2} \\ u_{r3} \end{bmatrix} \, dx \, dy \]

\[ = \frac{1}{2} \begin{bmatrix} r \\ r' \end{bmatrix}^T \int_A \begin{bmatrix} \sigma_{31}^0 I & \sigma_{32}^0 I & \sigma_{33}^0 I \\ \sigma_{31}^0 I & \sigma_{32}^0 I & \sigma_{33}^0 I \end{bmatrix} U \begin{bmatrix} r \\ r' \end{bmatrix} \]

\[ = \frac{1}{2} \begin{bmatrix} r \\ r' \end{bmatrix}^T D_0 g \begin{bmatrix} r \\ r' \end{bmatrix}, \tag{4.26} \]

where the initial stress strain energy density matrix \( D_0 \) is obtained after integration over the cross-sectional area as

\[ D_0 = \begin{bmatrix} \mu Q_1 & \mu^2 Q_1 & 0 & \mu^2 Q_1 \\ 0 & \mu Q_1 & 0 & \mu Q_1 \\ 0 & 0 & 0 & \mu Q_1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.27} \]

It is noted that the initial stresses in Eq. (4.27) relate to strains and displacements, hence, nonlinear displacements have been considered here. The terms involving initial torque \( M_3 \) are slightly different from those of some authors and discussions can be found in [127, 128, 131, 132]. The total strain energy density per unit length \( U_0 \) is the sum of the strain energy density due to initial stresses \( U_{\sigma} \) represented by \( D_0 \) and \( U_{\varepsilon} \) that due to linear strain by \( D_1 \) where
\[
U_e = \frac{1}{2} \left( \mathbf{r}' \right)^T \mathbf{b}^T Yb \mathbf{r} = \frac{1}{2} \left( \mathbf{r}' \right)^T \mathbf{D}_s \left( \mathbf{r}' \right),
\]

where \( \mathbf{b} \) is from Eq. (4.18), and

\[
\mathbf{D}_s = \begin{bmatrix}
\mu^2 GA \\
0 & \mu^2 GA \\
0 & 0 & 0 & \text{sym} \\
-\mu GA & 0 & 0 & GA + EI_0 \mu^2 \\
0 & -\mu GA & 0 & 0 & GA + EI_0 \mu^2 \\
0 & 0 & 0 & 0 & 0 \\
0 & \mu GA & 0 & 0 & -GA & 0 & GA \\
-\mu GA & 0 & 0 & GA & 0 & 0 & 0 & GA \\
0 & 0 & 0 & 0 & 0 & 0 & EA \\
0 & 0 & 0 & 0 & I_1 E \mu & 0 & 0 & 0 & EI_1 \\
0 & 0 & 0 & -I_2 E \mu & 0 & 0 & 0 & 0 & 0 & EI_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & GJ
\end{bmatrix}
\]

The higher order terms of \( \mu^2 \) are not negligible and must be retained in numerical simulation. For the convenience of numerical simulation, the non-dimensional equations are given by Eq. (4.30),

\[
\bar{\mu} = \mu l, \; \bar{g} = \frac{GA l^2}{EI_1}, \; \bar{r}_1 = \frac{l_1}{Al^2}, \; \bar{r}_2 = \frac{l_2}{Al^2}, \; \bar{r}_0 = \frac{l_0}{Al^2}, \; \bar{r}_3 = \frac{J}{Al^2}, \\
\Omega^4 = \omega^2 \rho A l^4 \; \bar{Q}_1 = \frac{Q_1}{EI_1}, \; \bar{Q}_2 = \frac{Q_2}{EI_1}, \; \bar{Q}_3 = \frac{Q_3}{EI_1}, \; \bar{m}_1 = \frac{M_1}{EI_1}, \; \bar{m}_2 = \frac{M_2}{EI_1}, \; \bar{m}_3 = \frac{M_3}{EI_1}, \; \bar{m}_4 = \frac{M_4}{EJ}.
\]

And the non-dimensional matrices of \( \mathbf{D}_s \), \( \mathbf{D}_m \) and \( \mathbf{D}_g \) are given in the following Eq. (4.31-4.33), respectively:
$$\mathbf{d}_i = \frac{EI_1}{l^2} \begin{bmatrix} \bar{\mu}^2 g \\ 0 \quad \bar{\mu}^2 g \\ 0 \quad 0 \quad 0 \\ -\bar{\mu}g & 0 & 0 & g + \bar{\mu}^2 r_i/r_i \\ 0 & -\bar{\mu}g & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \bar{\mu}g & 0 & 0 \\ -\bar{\mu}g & 0 & 0 & g + \bar{\mu}^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -g \quad 0 \quad g \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\bar{\mu}r_2/r_i & 0 & 0 & 0 \quad 0 \quad r_2/r_i \\ 0 & 0 & 0 & 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad g r_2 \end{bmatrix}$$ (4.31)

$$\mathbf{d}_m = \frac{EI_1}{l^2} \begin{bmatrix} 1 \\ 1 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix}.$$ (4.32)

$$\mathbf{d}_q = \frac{EI_1}{l^2} \begin{bmatrix} \bar{\mu}^2 q_1 \\ 0 \quad \bar{\mu}^2 q_3 \\ 0 \quad 0 \quad 0 \\ 0 \quad 0 \quad 0 \quad 2\bar{\mu}m_2 r_2/r_i + \bar{\mu}^2 q r \quad 2\bar{\mu}m_2 + \bar{\mu}^2 q r_i \\ 0 \quad 0 \quad 0 \quad 0 \\ -\bar{\mu}q_2 & 0 & 0 & 0 \\ -\bar{\mu}q_1 & 0 & 0 & 0 \\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad q_i \quad 0 \quad q_i \quad q_i \\ 0 \quad 0 \quad 0 \quad q_i - \bar{\mu}m_2 & 0 \quad 0 \quad 0 \quad q_i \\ 0 \quad 0 \quad 0 \quad 0 \quad m_i + \bar{\mu}q r_i & 0 \quad 0 \quad 0 \quad m_i \quad q r_i \\ 0 \quad 0 \quad 0 \quad -m_2 - \bar{\mu}q r_i & 0 \quad 0 \quad 0 \quad m_2 \quad 0 \quad q r_i \quad 0 \quad 0 \quad q r_i \\ \bar{\mu}m_2 & -\bar{\mu}m_2 & 0 & 0 \quad 0 \quad 0 \quad m_2 \quad 0 \quad q r_i \quad q r_i \end{bmatrix}.$$ (4.33)
4.5  $p$-elements for Pre-twisted Straight Beams

4.5.1  Shape functions

The generalized nodal displacements of the pre-twisted straight beam can be obtained from Eq. (4.14),

$$\mathbf{q}_g = \begin{bmatrix} v_1^b & v_2^b & v_3^b & \theta_1^b & \theta_2^b & \theta_3^b & v_1' & v_2' & v_3' & \theta_1' & \theta_2' & \theta_3' \end{bmatrix}^T,$$

where the superscript ‘$b$’ and ‘$t$’ means the displacements on the bottom (negative) end and the top (positive) end of the beam according to finite element convention.

The displacement $\mathbf{u}$ of a beam subject to constant external forces $Q_1$, $Q_2$, $Q_3$, $M_1$, $M_2$ and $M_3$ is given by

$$\mathbf{u} = N \mathbf{q},$$

where $N$ is the shape functions for the pre-twisted straight beam, $\mathbf{q} = \begin{bmatrix} \mathbf{q}_g \\ \mathbf{q}_i \end{bmatrix}$, $\mathbf{q}_g$ is the generalized nodal displacements defined in Eq.(4.34) and $\mathbf{q}_i$ is the internal freedoms of the beams.

$$\mathbf{q}_i = \begin{bmatrix} t_{v_1} & t_{v_2} & t_{v_3} & t_{\theta_1} & t_{\theta_2} & t_{\theta_3} \end{bmatrix}^T.$$

The shape functions satisfying the displacement continuity at element interface for a pre-twisted straight beam are as following:
Chapter 4: Dynamic stability of pre-twisted straight beams

\[ N(\xi) = \begin{bmatrix}
N_{x_1} \\
N_{x_2} \\
N_{x_3} \\
N_{\xi_1} \\
N_{\xi_2} \\
N_{\xi_3}
\end{bmatrix} = \begin{bmatrix}
f_i(\xi) & 0 & 0 & 0 & 0 & 0 \\
0 & f_i(\xi) & 0 & 0 & 0 & 0 \\
0 & 0 & f_i(\xi) & 0 & 0 & 0 \\
0 & 0 & 0 & f_i(\xi) & 0 & 0 \\
0 & 0 & 0 & 0 & f_i(\xi) & 0 \\
0 & 0 & 0 & 0 & 0 & f_i(\xi)
\end{bmatrix}, \quad (4.37) \]

where \( f_i(\xi) \) is the \( C^0 \) polynomial series which has been used by Houmat [13] giving in Appendix 1, and \( \xi \) is mapped from \((-1,1)\) to \((0,1)\), then \( \xi = \frac{x_3}{l} \) \((0 < \xi < 1)\) is the non-dimensional length, and \( l \) is the length of beam, and \( i = 1, 2, \ldots, p+2 \), \( p \) is the number of internal degrees of freedom.

### 4.5.2 Stiffness, mass and geometric matrices

The equation of the eigenvalue problem of the pre-twisted straight beams is

\[
\left( K - \omega^2 M - Q_1 G_{\xi_1} - Q_2 G_{\xi_2} - Q_3 G_{\xi_3} - M_1 G_{M_1} - M_2 G_{M_2} - M_3 G_{M_3} \right) q = 0, \quad (4.38)
\]

where \( Q_1, Q_2, Q_3, M_1, M_2, M_3 \) are the external forces. Then the stiffness, mass and geometric stiffness matrices can be obtained as follows:

\[
K = \sum_{e} K^e, \quad M = \sum_{e} M^e, \quad G = \sum_{e} G^e, \quad (4.39)
\]

where

\[
K^e = \frac{1}{2} \int_0^l \left( N \right)^T d(N)/dx_3 \left( N \right) dx_3, \quad (4.40a)
\]

\[
M^e = \frac{1}{2} \int_0^l N^T D_m N dx_3, \quad (4.40b)
\]
\[
G^e = \frac{1}{2} \int_0^l F_{QM} \left( \frac{d(N)}{dx_3} \right)^T D_x \left( \frac{d(N)}{dx_3} \right) dx_3,
\]  
(4.40c)

where \(F_{QM}\) represents the internal forces due to external forces \(Q_1, Q_2, Q_3, M_1, M_2\) or \(M_3\).

By non-dimensional, the equation of the eigenvalue problem of a pre-twisted straight beam given in Eq. (4.39) becomes

\[
\left( \bar{K} - \Omega^4 \bar{M} - q_1 \bar{G}_{g_1} - q_2 \bar{g}_{g_2} - q_3 \bar{G}_{g_3} - m_1 \bar{G}_{m_1} - m_2 \bar{G}_{m_2} - m_3 \bar{G}_{m_3} \right) \bar{q} = 0,
\]  
(4.41)

where

\[
\bar{K}^e = \frac{1}{2} \int_0^l \left( \frac{N(\xi)}{d[N(\xi)]/d\xi} \right)^T d_\xi \left( \frac{N(\xi)}{d[N(\xi)]/d\xi} \right) d\xi,
\]  
(4.42a)

\[
\bar{M}^e = \frac{1}{2} \int_0^l N^T(\xi) d_m N(\xi) d\xi,
\]  
(4.42b)

\[
\bar{G}^e = \frac{1}{2} \int_0^l F_{QM} \left( \frac{d[N(\xi)]}{d\xi} \right)^T d_\xi \left( \frac{N(\xi)}{d[N(\xi)]/d\xi} \right) d\xi,
\]  
(4.42c)

where the overbar denotes the non-dimensional matrices of \(K^e, M^e, G^e_{g_1}, G^e_{g_2}, G^e_{g_3}, G^e_{m_1}, G^e_{m_2}\) and \(G^e_{m_3}\), non-dimensional displacement vector \(q\), and non-dimensional internal forces \(F_{QM}\).

For a pre-twisted straight beam clamped on one end and free on the other end and
is subject only to axial loads \( Q_3 \) and/or torque \( M_3 \) on the free end, the internal forces will be constant along the length of the beam. However, when the beam is subject to end shears \( Q_1 \) or \( Q_2 \) or end moments \( M_1 \) or \( M_2 \), since the beam is pre-twisted, the axes 1 and 2 of the current plane has a twisted angle \( \mu \bar{z} \) with the free-end, therefore, the internal forces become functions of \( \mu \bar{z} \). The internal forces due to end forces \( Q_1, Q_2, M_1 \) or \( M_2 \) along the length of the beam are given in equations (4.43a-d), respectively.

\[
\begin{align*}
\tilde{Q}_1 \neq 0, \tilde{Q}_2 = 0, \tilde{M}_1 = 0, \tilde{M}_2 = 0 & \quad \tilde{Q}_1 = 0, \tilde{Q}_2 \neq 0, \tilde{M}_1 = 0, \tilde{M}_2 = 0 \\
Q_1 = \tilde{Q}_1 \cos(\mu \bar{z}), & \quad Q_1 = \tilde{Q}_1 \sin(\mu \bar{z}), \\
Q_2 = -\tilde{Q}_2 \sin(\mu \bar{z}), & \quad Q_2 = \tilde{Q}_2 \cos(\mu \bar{z}), \\
M_1 = \tilde{M}_1 \bar{z} \sin(\mu \bar{z}), & \quad M_1 = -\tilde{M}_2 \bar{z} \cos(\mu \bar{z}), \\
M_2 = \tilde{M}_2 \bar{z} \cos(\mu \bar{z}), & \quad M_2 = \tilde{M}_2 \bar{z} \sin(\mu \bar{z}),
\end{align*}
\]

\[
\begin{align*}
\tilde{Q}_1 = 0, \tilde{Q}_2 = 0, \tilde{M}_1 \neq 0, \tilde{M}_2 = 0 & \quad \tilde{Q}_1 = 0, \tilde{Q}_2 = 0, \tilde{M}_1 \neq 0, \tilde{M}_2 \neq 0 \\
Q_1 = 0, & \quad Q_1 = 0, \\
Q_2 = 0, & \quad Q_2 = 0, \\
M_1 = \tilde{M}_1 \cos(\mu \bar{z}), & \quad M_1 = \tilde{M}_2 \sin(\mu \bar{z}), \\
M_2 = -\tilde{M}_1 \sin(\mu \bar{z}), & \quad M_2 = \tilde{M}_2 \cos(\mu \bar{z}),
\end{align*}
\]

where the curly overbar ‘\( \sim \)’ on \( Q_1 \), \( Q_2 \), \( M_1 \) or \( M_2 \) denotes the external forces applied on the free end of the cantilevered straight beam and \( \bar{z} = l - x_3 = l (1 - \xi) \) then \( \mu \bar{z} = \mu (1 - \xi) \).

The coefficients of the stiffness matrix \( \widetilde{K}^e \), mass matrix \( \widetilde{M}^e \) and geometric stiffness matrices \( \widetilde{G}^e_{q_1}, \widetilde{G}^e_{q_2}, \widetilde{G}^e_{q_3}, \widetilde{G}^e_{m_1}, \widetilde{G}^e_{m_2} \) and \( \widetilde{G}^e_{m_3} \) are given in Appendix 5.
4.6 Numerical Examples

4.6.1 Results comparison

4.6.1.1 Natural frequency convergence study and compared with existed solutions

Considering a cantilevered beam has a length of \( l = 3.048 \text{m} \) with a pre-twist angle \( \bar{\mu} = \frac{2\pi}{9} \) as the first example [152, 157, 158]. The properties of the beam are: \( E = 70 \times 10^9 \text{N/m}^2, \ G = 27 \times 10^9 \text{N/m}^2, \ \rho = 2700 \text{kg/m}^3, \ A = 0.0127667 \text{m}^2, \ EI_x = 2869.7 \text{N·m}^2, \ EI_z = 57393 \text{N·m}^2. \) The present frequencies \( \omega (\text{rad/s}) \) are compared with Rosen [152], Banerjee [157-158] in Table 1. It is found that when taking shear effective factor as \( k = 1/1.2, \) the polynomial terms \( p = 12, \) the present results fit well with Banerjee’s Timoshenko theory results. Therefore, the polynomial terms \( p = 12 \) will be chosen for the study in this chapter.

<table>
<thead>
<tr>
<th>Number of polynomial terms</th>
<th>Mode number</th>
<th>( \omega ) (rad/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present ( (p=12) ) ( (k=1) )</td>
<td>1</td>
<td>3.4718</td>
</tr>
<tr>
<td>Present ( (p=4) ) ( (k=1/1.2) )</td>
<td>1</td>
<td>3.4722</td>
</tr>
<tr>
<td>Present ( (p=6) ) ( (k=1/1.2) )</td>
<td>1</td>
<td>3.4718</td>
</tr>
<tr>
<td>Present ( (p=8) ) ( (k=1/1.2) )</td>
<td>1</td>
<td>3.4718</td>
</tr>
<tr>
<td>Present ( (p=10) ) ( (k=1/1.2) )</td>
<td>1</td>
<td>3.4718</td>
</tr>
<tr>
<td>Present ( (p=12) ) ( (k=1/1.2) )</td>
<td>1</td>
<td>3.4718</td>
</tr>
<tr>
<td>Rosen [152]</td>
<td></td>
<td>3.4726</td>
</tr>
</tbody>
</table>

4.6.1.2 Comparison with results of ANSYS

Consider a clamped free beam with rectangular solid section as shown in Figure 4.3 having the following properties:
Young’s modulus $E=200$ Mpa, shear modulus $G=E/2(1+\nu)$, and shear effective factor $k=1$, Poisson’s ratio $\nu=0.3$ and the mass density $\rho=7.8$, with length $l=1m$, cross-section $b=1/1.5\text{mm}$ and $d=1.5\text{mm}$, the area $A=bd=1\text{mm}^2$, the second moment of area $I_1=1/12bd^3$, $I_2=1/12bd^3$, and the polar moment of area $I_0=I_1+I_2=0.2245\text{mm}^4$, when $b\leq d$, the exact expressions of the torsion constant $J$ has been given by Conner [167], having the value of $0.1067\text{mm}^4$.

To show the accuracy of the present method, the natural frequency $\Omega$ and buckling loads $\sqrt{q_1}$, $\sqrt{q_2}$, $\sqrt{q_3}$ obtained by $p$-element are compared with the results of ANSYS [168] (choosing 100 BEAM188 elements in ANSYS) in Table 4.2, and good agreement is found. Since it is hard to apply pure bending moments or torque on beam element in ANSYS, the buckling moments (and torque) $\sqrt{m_1}$, $\sqrt{m_2}$, and $m_3$ obtained by $p$-element are not be able to compare with those of ANSYS. For a pre-twisted beam, the value of the buckling torque obtained in the same direction with the pre-twist angle is different from that in the opposite direction (negative pre-twist). Then, the buckling torque in the same direction with the pre-twist angle is defined as $m_3$ (+) and the opposite one is $m_3$ (-). Conventionally, torsion constant $J$ is taken equal to polar moment of area $I_0$ in beam vibration and buckling analysis. Actually, when the aspect ratio $d/b$ becomes larger for rectangular cross-sections, the value of $J$ has a great difference with $I_0$. 

Figure 4.3 The dimensions of the cantilevered beam
Therefore, the relative error by taken \((J=I_0)\) with \((J\neq I_0)\) for frequency \(\Omega\) and buckling load parameters \(\sqrt{q_1}, \sqrt{q_2}, \sqrt{q_3}, \sqrt{m_1}, \sqrt{m_2}\) and \(m_3\) are computed in Table 4.3. As shown in Table 4.3, the results by taking \((J=I_0)\) have a 21% difference with the accurate results by taking \((J\neq I_0)\) for the first six modes of shear buckling loads \(\sqrt{q_1}, \sqrt{q_2}\) and moments \(\sqrt{m_1}, \sqrt{m_2}\), but the difference between \(J\) and \(I_0\) has very little influence on the first six modes of axial loads \(\sqrt{q_3}\) and buckling torques \(m_3\). Therefore, in this chapter we take \((J\neq I_0)\) for more accurate solutions.

Table 4.2 Frequency \(\Omega\) and buckling loads load parameters \(\sqrt{q_1}, \sqrt{q_2}, \sqrt{q_3}\) of a pre-twisted clamped-free beam with \(l=1\text{m}, \ b=1/1.5\text{mm}, \ d=1.5\text{mm}, \ \bar{\mu} = \pi/2\) and compared with results of ANSYS

<table>
<thead>
<tr>
<th>Frequencies and Buckling loads</th>
<th>Mode number</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Omega)</td>
<td>1</td>
</tr>
<tr>
<td>ANSYS</td>
<td></td>
</tr>
<tr>
<td>(\sqrt{q_1})</td>
<td></td>
</tr>
<tr>
<td>ANSYS</td>
<td></td>
</tr>
<tr>
<td>(\sqrt{q_2})</td>
<td></td>
</tr>
<tr>
<td>ANSYS</td>
<td></td>
</tr>
<tr>
<td>(\sqrt{q_3})</td>
<td></td>
</tr>
<tr>
<td>ANSYS</td>
<td></td>
</tr>
</tbody>
</table>
Table 4.3 The relative error when taking \( J \neq I_0 \) and \( J = I_0 \) for frequency \( \Omega \) and buckling loads parameters, \( \sqrt{q_1}, \sqrt{q_2}, \sqrt{q_3}, \sqrt{m_1}, \sqrt{m_2} \) and \( m_3 \) of a pre-twisted clamped-free beam with \( l = 1 \text{m}, b = 1/1.5 \text{mm}, d = 1.5 \text{mm}, \mu = \pi/2 \)

<table>
<thead>
<tr>
<th>Frequencies and Buckling loads</th>
<th>Mode number</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Omega )</td>
<td>1</td>
</tr>
<tr>
<td>( \sqrt{q_1} )</td>
<td>0.0004</td>
</tr>
<tr>
<td>( \sqrt{q_2} )</td>
<td>20.4370</td>
</tr>
<tr>
<td>( \sqrt{m_1} )</td>
<td>0</td>
</tr>
<tr>
<td>( \sqrt{m_2} )</td>
<td>20.4370</td>
</tr>
<tr>
<td>( m_3 )</td>
<td>0</td>
</tr>
<tr>
<td>( m_3 ) (+)</td>
<td>0</td>
</tr>
</tbody>
</table>

4.6.2 The influence of rigidity ratio and pre-twisted angle

We are interested to find out the influence of the initial twist angle \( \mu \) on the static instability phenomena when the twisted beam is subjected to different external forces. The non-dimensional parameter \( \mu \) is taken varying from 0 to \( 2\pi \). Let the area \( A = bd \) to be constant equal to \( 1 \text{mm}^2 \) and the width \( d \) varied from 1mm to 1.4mm. And the shear effective factor is chosen as \( 1/1.2 \) in this section. The flexural rigidity ratios defined as \( r = I_2/I_1 \) varied with respect to the width \( d \) are given in Table 4.4. The influence of pre-twist angle \( \mu \) and flexural rigidity ratio \( r \) on the first two modes of natural vibration frequencies \( \Omega \) and buckling loads, including shear forces \( \sqrt{q_1} \) and \( \sqrt{q_2} \), axial force \( \sqrt{q_3} \) bending moments \( \sqrt{m_1}, \sqrt{m_2} \) and torque \( m_3 \) are tabulated in Tables 4.5a-h, respectively. Meanwhile, the first two modes of frequency and buckling loads varying with the initial twist angle are shown graphically in Figure 4.4(a-h), respectively. The solid lines and dotted lines as shown in Figure 4.4 represent the first modes and second modes of the frequency or buckling loads, respectively. The lines which marked by ‘upward
pointing triangle $\Delta$, ‘right-pointing triangle $\triangleright$’, ‘left-pointing triangle $\triangleleft$’, ‘square $\square$’ or ‘circle $\circ$’ corresponding to the frequency or buckling loads of beams with rigidity ratio $r, i = 1, 2, 3, 4, 5$, respectively. As shown in Figure 4.4(a-h), when there is no pre-twist angle, the results are compared well with existing literatures. The natural frequencies and/or buckling loads for rigidity ratio $r_1=1$ (i.e. square cross-sections beams) are kept constant with the twist angle. It is observed in Figure 4.4a and Figure 4.4b that the first modes of the frequency and axial loads are increasing with the pre-twist angle and the rigidity ratio while the second modes decreased. As shown in Figure 4.4d for $\sqrt{q_2}$ and Figure 4.4e for $\sqrt{m_1}$, both the first two buckling loads are increasing with the rigidity ratio monotonically. However, the buckling loads are not monotonically with respect to the pre-twist angle. Similar phenomenon can be found with $\sqrt{q_1}$ in Figure 4.4c and $\sqrt{m_2}$ in Figure 4.4f. The situations for the first two buckling modes of torque $m_3 (-)$ or $m_3 (+)$ become more complicated. It is observed that the torques are not affected monotonically neither by the rigidity ratio nor with the twist angle as shown in Table 4.5(g-h) and Figure 4.4(g-h).

<table>
<thead>
<tr>
<th>$d$ (mm)</th>
<th>1</th>
<th>1.1</th>
<th>1.2</th>
<th>1.3</th>
<th>1.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d/b$</td>
<td>1</td>
<td>1.21</td>
<td>1.44</td>
<td>1.69</td>
<td>1.96</td>
</tr>
<tr>
<td>$I_o$ (mm$^4$)</td>
<td>0.1667</td>
<td>0.1697</td>
<td>0.1779</td>
<td>0.1901</td>
<td>0.2059</td>
</tr>
<tr>
<td>$J$ (mm$^4$)</td>
<td>0.1406</td>
<td>0.1382</td>
<td>0.1323</td>
<td>0.1244</td>
<td>0.1156</td>
</tr>
<tr>
<td>$r=I_2/I_1$</td>
<td>1</td>
<td>1.646</td>
<td>2.0376</td>
<td>2.8561</td>
<td>3.8416</td>
</tr>
</tbody>
</table>
### Table 4.5a Influence of pre-twist angle $\mu$ and rigidity ratio $r$ on natural frequency $\Omega$

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>Mode 1</th>
<th>Mode 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$r_1$</td>
<td>$r_2$</td>
</tr>
<tr>
<td>0</td>
<td>1.8751</td>
<td>1.8751</td>
</tr>
<tr>
<td>$\pi/4$</td>
<td>1.8751</td>
<td>1.8773</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>1.8751</td>
<td>1.8834</td>
</tr>
<tr>
<td>$3\pi/4$</td>
<td>1.8751</td>
<td>1.8923</td>
</tr>
<tr>
<td>$\pi$</td>
<td>1.8751</td>
<td>1.9025</td>
</tr>
<tr>
<td>$5\pi/4$</td>
<td>1.8751</td>
<td>1.9122</td>
</tr>
<tr>
<td>$3\pi/2$</td>
<td>1.8751</td>
<td>1.9200</td>
</tr>
<tr>
<td>$7\pi/4$</td>
<td>1.8751</td>
<td>1.9255</td>
</tr>
<tr>
<td>$2\pi$</td>
<td>1.8751</td>
<td>1.9294</td>
</tr>
</tbody>
</table>

### Table 4.5b Influence of pre-twist angle $\mu$ and rigidity ratio $r$ on buckling loads $\sqrt{q_i}$

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>Mode 1</th>
<th>Mode 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$r_1$</td>
<td>$r_2$</td>
</tr>
<tr>
<td>0</td>
<td>1.5708</td>
<td>1.5708</td>
</tr>
<tr>
<td>$\pi/4$</td>
<td>1.5708</td>
<td>1.5769</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>1.5708</td>
<td>1.5943</td>
</tr>
<tr>
<td>$3\pi/4$</td>
<td>1.5708</td>
<td>1.6192</td>
</tr>
<tr>
<td>$\pi$</td>
<td>1.5708</td>
<td>1.6461</td>
</tr>
<tr>
<td>$5\pi/4$</td>
<td>1.5708</td>
<td>1.6674</td>
</tr>
<tr>
<td>$3\pi/2$</td>
<td>1.5708</td>
<td>1.678</td>
</tr>
<tr>
<td>$7\pi/4$</td>
<td>1.5708</td>
<td>1.6817</td>
</tr>
<tr>
<td>$2\pi$</td>
<td>1.5708</td>
<td>1.6850</td>
</tr>
</tbody>
</table>

### Table 4.5c Influence of pre-twist angle $\mu$ and rigidity ratio $r$ on buckling loads $\sqrt{q_i}$

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>Mode 1</th>
<th>Mode 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$r_1$</td>
<td>$r_2$</td>
</tr>
<tr>
<td>0</td>
<td>1.7177</td>
<td>1.794</td>
</tr>
<tr>
<td>$\pi/4$</td>
<td>1.7177</td>
<td>1.8246</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>1.7177</td>
<td>1.8926</td>
</tr>
<tr>
<td>$3\pi/4$</td>
<td>1.7177</td>
<td>1.9345</td>
</tr>
<tr>
<td>$\pi$</td>
<td>1.7177</td>
<td>1.9116</td>
</tr>
<tr>
<td>$5\pi/4$</td>
<td>1.7177</td>
<td>1.8700</td>
</tr>
<tr>
<td>$3\pi/2$</td>
<td>1.7177</td>
<td>1.8559</td>
</tr>
<tr>
<td>$7\pi/4$</td>
<td>1.7177</td>
<td>1.8666</td>
</tr>
<tr>
<td>$2\pi$</td>
<td>1.7177</td>
<td>1.8783</td>
</tr>
</tbody>
</table>
Table 4.5d Influence of pre-twist angle $\mu$ and rigidity ratio $r$ on buckling loads $\sqrt{q_i}$

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>Mode 1</th>
<th>Mode 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$r_1$</td>
<td>$r_2$</td>
</tr>
<tr>
<td>$0$</td>
<td>1.7177</td>
<td>1.9734</td>
</tr>
<tr>
<td>$\pi/4$</td>
<td>1.7177</td>
<td>1.9285</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>1.7177</td>
<td>1.8533</td>
</tr>
<tr>
<td>$3\pi/4$</td>
<td>1.7177</td>
<td>1.8198</td>
</tr>
<tr>
<td>$\pi$</td>
<td>1.7177</td>
<td>1.8371</td>
</tr>
<tr>
<td>$5\pi/4$</td>
<td>1.7177</td>
<td>1.8737</td>
</tr>
<tr>
<td>$3\pi/2$</td>
<td>1.7177</td>
<td>1.8899</td>
</tr>
<tr>
<td>$3\pi/4$</td>
<td>1.7177</td>
<td>1.8786</td>
</tr>
<tr>
<td>$2\pi$</td>
<td>1.7177</td>
<td>1.8671</td>
</tr>
</tbody>
</table>

Table 4.5e Influence of pre-twist angle $\mu$ and rigidity ratio $r$ on buckling loads $\sqrt{m_i}$

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>Mode 1</th>
<th>Mode 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$r_1$</td>
<td>$r_2$</td>
</tr>
<tr>
<td>$0$</td>
<td>1.0747</td>
<td>1.2347</td>
</tr>
<tr>
<td>$\pi/4$</td>
<td>1.0747</td>
<td>1.1965</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>1.0747</td>
<td>1.1456</td>
</tr>
<tr>
<td>$3\pi/4$</td>
<td>1.0747</td>
<td>1.1412</td>
</tr>
<tr>
<td>$\pi$</td>
<td>1.0747</td>
<td>1.1715</td>
</tr>
<tr>
<td>$5\pi/4$</td>
<td>1.0747</td>
<td>1.1874</td>
</tr>
<tr>
<td>$3\pi/2$</td>
<td>1.0747</td>
<td>1.1718</td>
</tr>
<tr>
<td>$7\pi/4$</td>
<td>1.0747</td>
<td>1.1615</td>
</tr>
<tr>
<td>$2\pi$</td>
<td>1.0747</td>
<td>1.1718</td>
</tr>
</tbody>
</table>

Table 4.5f Influence of pre-twist angle $\mu$ and rigidity ratio $r$ on buckling loads $\sqrt{m_i}$

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>Mode 1</th>
<th>Mode 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$r_1$</td>
<td>$r_2$</td>
</tr>
<tr>
<td>$0$</td>
<td>1.0747</td>
<td>1.1225</td>
</tr>
<tr>
<td>$\pi/4$</td>
<td>1.0747</td>
<td>1.1495</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>1.0747</td>
<td>1.2009</td>
</tr>
<tr>
<td>$3\pi/4$</td>
<td>1.0747</td>
<td>1.2071</td>
</tr>
<tr>
<td>$\pi$</td>
<td>1.0747</td>
<td>1.1715</td>
</tr>
<tr>
<td>$5\pi/4$</td>
<td>1.0747</td>
<td>1.1567</td>
</tr>
<tr>
<td>$3\pi/2$</td>
<td>1.0747</td>
<td>1.1718</td>
</tr>
<tr>
<td>$7\pi/4$</td>
<td>1.0747</td>
<td>1.1825</td>
</tr>
<tr>
<td>$2\pi$</td>
<td>1.0747</td>
<td>1.1718</td>
</tr>
</tbody>
</table>
### Table 4.5g Influence of pre-twist angle $\mu$ and rigidity ratio $r$ on buckling loads $m_3 (-)$

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>Mode 1</th>
<th>Mode 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$r_1$</td>
<td>$r_2$</td>
</tr>
<tr>
<td>0</td>
<td>1.5708</td>
<td>1.3599</td>
</tr>
<tr>
<td>$\pi/4$</td>
<td>1.5708</td>
<td>1.4483</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>1.5708</td>
<td>1.5412</td>
</tr>
<tr>
<td>$3\pi/4$</td>
<td>1.5708</td>
<td>1.4946</td>
</tr>
<tr>
<td>$\pi$</td>
<td>1.5708</td>
<td>1.4826</td>
</tr>
<tr>
<td>$5\pi/4$</td>
<td>1.5708</td>
<td>1.5037</td>
</tr>
<tr>
<td>$3\pi/2$</td>
<td>1.5708</td>
<td>1.5417</td>
</tr>
<tr>
<td>$7\pi/4$</td>
<td>1.5708</td>
<td>1.5154</td>
</tr>
<tr>
<td>$2\pi$</td>
<td>1.5708</td>
<td>1.5063</td>
</tr>
</tbody>
</table>

### Table 4.5h Influence of pre-twist angle $\mu$ and rigidity ratio $r$ on buckling loads $m_3 (+)$

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>Mode 1</th>
<th>Mode 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$r_1$</td>
<td>$r_2$</td>
</tr>
<tr>
<td>0</td>
<td>1.5708</td>
<td>1.3599</td>
</tr>
<tr>
<td>$\pi/4$</td>
<td>1.5708</td>
<td>1.3077</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>1.5708</td>
<td>1.3007</td>
</tr>
<tr>
<td>$3\pi/4$</td>
<td>1.5708</td>
<td>1.3339</td>
</tr>
<tr>
<td>$\pi$</td>
<td>1.5708</td>
<td>1.3961</td>
</tr>
<tr>
<td>$5\pi/4$</td>
<td>1.5708</td>
<td>1.4728</td>
</tr>
<tr>
<td>$3\pi/2$</td>
<td>1.5708</td>
<td>1.5385</td>
</tr>
<tr>
<td>$7\pi/4$</td>
<td>1.5708</td>
<td>1.4890</td>
</tr>
<tr>
<td>$2\pi$</td>
<td>1.5708</td>
<td>1.4847</td>
</tr>
</tbody>
</table>
Chapter 4: Dynamic stability of pre-twisted straight beams

(a) Natural frequency $\Omega$

(b) Buckling loads $\sqrt{q_3}$

(c) Buckling loads $\sqrt{q_1}$

(d) Buckling loads $\sqrt{q_2}$
Chapter 4: Dynamic stability of pre-twisted straight beams

(e) Buckling loads $\sqrt{m_1}$

(f) Buckling loads $\sqrt{m_2}$

(g) Buckling loads $m_3(-)$

(h) Buckling loads $m_3(+)$. 
4.6.3 Interaction of natural frequency and buckling loads

Consider the dynamic buckling of a rectangular solid section column clamped at one end and free at the other end having the properties along the principal axes for Poisson's ratio 0.3 and effective shear area \( k = 1/1.2 \) as in Section 4.6.2. The beam has a width \( d = 1.2 \) mm and \( b = 1/1.2 \) mm, with the rigidity ratio \( r = 1.24 \). The interactions of natural frequency and buckling loads under the influence of the angle of pre-twist \( \alpha = (i - 1) \pi / 4, i = 1, 2, \ldots, 9 \) are considered. In this section, the first, second and third modes of the interaction diagrams are distinguished by solid-lines, dotted-lines and dashed-lines respectively.

4.6.3.1 Interaction of natural frequency and buckling compression

The interaction of natural frequency and buckling compression under the influence of the angle of twist is computed in Figures 4.5(a) and 4.5(b). Figure 4.5(a) shows the variation of the dynamic buckling loads with the twist angle and Figure 4.5(b) shows the top view. The numbers of the twist angle \( \alpha = (i - 1) \pi / 4 \), where \( i \) varied from 1 to 9 are marked in Figure 4.5b. The first two natural frequencies for the un-twisted beam are 1.8751 and \( 1.8751 \times r^{\pi/4} = 2.2501 \) and the first two buckling compressions are \( \pi/2 = 1.5708 \) and \( 1.5708 \times r^{\pi/2} = 2.2619 \) as
expected. It is interesting to note that the first mode increases together with the angle of twist but the second decreases seeming to close the first two modes together. The gaps close monotonically as the angle of twist increases for natural frequencies and buckling compressions.

Figure 4.5 The influence of the twist angle on interaction diagram of natural frequencies and buckling compressions
4.6.3.2 Interaction of natural frequency and buckling torques

The interaction of natural frequency and buckling torque under the influence of the pre-twist angle is computed in Figures 4.6(a) and 4.6(b). Figure 4.6(a) shows the variation of the dynamic buckling torques with the twist angle and Figure 4.6(b) shows the top view where the twist angle is numbered. The first two natural frequencies for the un-twisted beam are 1.8751 and 2.2501 as in Section 6.3.1 and the first two buckling torques are 1.1570 and 1.7867 (comparing to 1.1570 and 1.7867 for Euler assumption [132]) as expected. It is noted that the two frequency modes increases together with the angle of twist. However, unlike natural frequencies and compressions, the closeness is not monotonic for buckling torque, where the gaps are smallest near the twist angle $\mu = 3\pi/2$. Similar phenomenon can be found for the buckling torque $m_3 (+)$ shown in Figure 4.7(a-b).
Figure 4.6 The influence of the angle of twist on interaction diagram of natural frequencies and buckling torques $m_3 (-)$
4.6.3.4 Interaction of natural frequency and shear buckling loads $\sqrt{q_1}$ and moments $\sqrt{m_2}$

Since the internal forces along the length of the pre-twist beam due to loads $\sqrt{q_1}$ or $\sqrt{m_2}$ are very similar, the interaction of natural frequency and shear buckling loads $\sqrt{q_1}$ or $\sqrt{m_2}$ under the influence of the pre-twist angle are considered together in this section, as shown in Figures 4.8-4.9, respectively. The variation of the dynamic buckling loads with the twist angle and the top view are shown as the previous sections. When there is no pre-twist, the straight beam will only buckle about axis 1 under shear load $\sqrt{q_1}$ as shown in Figure 4.8 or moment $\sqrt{m_2}$ as shown in Figure 4.9. The first vibration mode begins at the point (frequency $\lambda$=1.8751, loads $\sqrt{q_1}$ or $\sqrt{m_2}$ =0), and ends at the buckling mode ($\lambda$=0 and
\[ \sqrt{q_1} = 1.8534, \text{ or } \sqrt{m_2} = 1.1596 \] (as expected in Timoshenko [135])). Since the beam will not buckle about axis 2, the second vibration mode beginning at the point \((\lambda=2.2501, \sqrt{q_1} \text{ or } \sqrt{m_2} = 0)\) is a straight line vertical to the frequency axis. When a pre-twist angle exists, the case \(\bar{\mu} = \pi/4\) \((i=2)\) is taken as an example to compare with \(\bar{\mu} = 0, \,(i=1)\) as shown in Figure 4.8(c). To show the first four frequency modes, the frequency axis is extended from \((0, 2.5)\) given in Figure 4.8(b) to \((0, 6)\) as shown in Figure 4.8(c). When \(\bar{\mu} = \pi/4\) the beam will buckle about axis 1, beginning at the first frequency mode value \(\Omega=1.8786\), and ending at the first buckling loads mode value \(\sqrt{q_1} = 1.9064\). Meanwhile, the beam will also buckle about axis 2, corresponding to the second frequency mode value \(\Omega=2.2412\) and the second buckling mode loads \(\sqrt{q_1} = 3.0709\). While for a no pre-twist beam, it is the third frequency mode \(\Omega=4.6941\) rather than the second mode \((\Omega=2.2501)\) to buckle at the second buckling mode load \(\sqrt{q_1} = 2.9617\). And as shown in Figure 4.8(b), although the buckling capacity for the pre-twisted beams is not increased monotonically with the twist angle, the buckling capacity is enhanced when compared with a no pre-twist beam. It is interesting to note that when \(\bar{\mu} = 7\pi/4, \,(i=8)\) i.e., the lines numbered as the 8\(^{th}\) lines in Figure 4.8(b) for \(\sqrt{q_1}\) or Figure 4.9(b) for \(\sqrt{m_2}\), the interaction diagrams beginning at the first frequency mode seems to end at the second buckling mode, and the lines starting at the second frequency mode will finally buckle at the first buckling mode, a more detail view can be seen from the lines numbered as the 8\(^{th}\) lines given in Figure 4.8(c).
Figure 4.8 The influence of the angle of twist on interaction diagram of natural frequency and shear buckling loads $\sqrt{q_1}$.
4.6.3.5 Interaction of natural frequency and shear buckling loads \( \sqrt{q_2} \) and moments \( \sqrt{m_1} \)

In this example, the interaction of natural frequency and shear buckling loads \( \sqrt{q_2} \) or \( \sqrt{m_1} \) under the influence of the angle of twist are computed in Figures 4.10-4.11, respectively. When there is no pre-twist, the beam will only buckle about axis 2, starting at the frequency point 2.2501. While when once a pre-twist is presented, the first buckling mode of the interaction diagram will start at frequency point 1.8751. Unlike the cases for buckling loads \( \sqrt{q_1} \) or \( \sqrt{m_2} \), the buckling capacity of the beam under loads \( \sqrt{q_2} \) or \( \sqrt{m_1} \) are weakened by the pre-twist angle. When \( \pi = 5\pi/4 \) numbered as the 6th lines for interaction diagram of frequency with buckling load \( \sqrt{q_2} \) or buckling moment \( \sqrt{m_1} \) as
shown in Figure 4.10(b) and Figure 4.11(b), an avoid crossing phenomenon appears which indicates that the interaction diagram beginning at the first frequency mode will finally buckle at the second buckling mode, and the interaction diagram starting at the second frequency mode will end at the first buckling mode.

Figure 4.10 The influence of the angle of twist on interaction diagram of natural frequency and shear buckling loads $\sqrt{q_2}$
Figure 4.11 The influence of the angle of twist on the interaction diagram of
natural frequency and buckling moments $\sqrt{m_1}$
4.7 Conclusion

The influence of all kinds of initial stresses due to compression, shears, moments
and torque on the natural vibration of pre-twisted straight beam based on the
Timoshenko theory has been formulated successfully with the use of
three-dimensional Green strain tensor. The Frenet formulae in differential
geometry were employed to treat the pre-twist. The governing equations and the
associated natural boundary conditions were derived from the variational principle.
It is noted that the first two modes tend to close together as the angle of twist
increases. The gaps reduce monotonically as the angle of twist increases. However,
unlike natural frequencies and compressions, the closeness is not monotonic for
buckling shears, moments and torques.
CHAPTER 5

FOURIER $p$-ELEMENTS FOR DYNAMIC STABILITY PROBLEMS OF MINDLIN PLATES

An analytically integrated trapezoidal Fourier $p$-element has been introduced by Leung and Zhu to solve transverse vibration problems of Mindlin plates. In their studies trigonometric functions are used as enriching shape functions to avoid the ill-conditioning problems associated with high order polynomials in a conventional $p$-element. The trapezoidal $p$-elements are analytically integrated in closed form. This chapter extends the trapezoidal Fourier $C^0$ $p$-elements to study the buckling problems and dynamic stability problems of Mindlin single plates and plate systems with different shapes and various boundary conditions. The interaction diagrams of skew and trapezoidal plates subject to uniaxial compression will be given. Since polygons can be discretized as combinations of trapezoidal and rectangular elements. Problems of polygonal plates subject to isotropic in-plane compressive loads are discussed in details. Moreover, the relationship of frequency and load for Mindlin plate systems composed of plate elements with different thicknesses will be considered in this chapter.
5.1 Introduction

Much review and work on vibration problems of Mindlin plates has been done by Leung and Zhu using analytically integrated Fourier $p$-elements [92] and polynomial $p$-elements [94, 95, 97]. In this section only literatures on buckling problems of Mindlin plates are reviewed. It is well-known that when the thickness of the plate increases, the shear effects will significantly influence the results of the buckling analysis. In this case, the Mindlin plate theory [169] is required to take into account the first order shear deformation effect. Over the past decades, the investigation of buckling problems of non-rectangular or non-circular thick plates becomes a main topic. The Rayleigh-Ritz method was firstly used to solve the buckling problem of thick skew plates by Kitipornchai and Xiang [170]. They considered skew plates subjected to uniaxial compression with simply supported and clamped boundary conditions. Xiang and Wang [171] extended this method to study skew plates subject to in-plane shear loadings. Mesh free method [172] was also used to solve rectangular plates and skew plates. A discrete singular convolution method had been used by Omer [173] to solve buckling analyses of composite skew plates with various boundary conditions.

However, the examples considered in the literatures above were limited to rectangular plates or skew plates where the in-plane stresses in the plate are constant inside the whole plate. Early studies of buckling problems of simply supported thin plates tapered in plane-form were given by Klein [174] using the method of collocation. The trapezoidal plates which are loaded in compression along the parallel edges, and shear forces along the sloping edges of the given plate were considered. Herdi and Tutuncu [175] used finite element method to investigate composite trapezoidal plates with uniaxial loads on the parallel sides.
In their study, the in-plane compression force was linearly varying with respect to the height of the trapezoidal plate. Buckling of trapezoidal thin plates with different geometries was studied by Saadatpour et al [176] using the Galerkin method.

In a $p$-version element, the trigonometric functions are more effective in predicting the medium- and high-frequency modes than the Legendre orthogonal polynomials both in precision and in avoiding the ill-conditioning problems [84]. Leung and Zhu [93-95] adopted an analytically integrated Fourier $p$-version element to overcome this problem. And this element has been used to obtain the frequency parameters of Mindlin plates with different shapes and boundary conditions. The mainly purpose of this chapter is to extend Fourier $p$-elements to buckling problems and dynamic stability problems of Mindlin plates. A conference paper is presented by the author and Leung [177] to solve the linear buckling problems of Mindlin plates using polynomial $p$-elements. In that paper the buckling intensity factors of skew plates subject to uniaxial compression or in-plane shear loads, and trapezoidal plates under uniaxial compression on two parallel sides or on two unparallel sides are considered. Another conference paper published by Leung and the author [178] considered the effects of concentrated of initial stresses on global buckling of plates, and the results are compared with results of ANSYS. Since the Fourier $p$-elements are more effective for medium and high modes of the structures when compared with those of polynomial $p$-elements, in this chapter Fourier $C^0$ $p$-element are used to investigate the dynamic stability problems of Mindlin plates. The governing equations of Mindlin plates are given in Section 5.2. The formulation of the Fourier $p$-elements for Mindlin plates is briefly discussed in Section 5.3 and numerical examples are given in Section 5.4. The interaction influence between natural frequencies and
buckling loads are discussed in detail.

## 5.2 Governing Equations of Mindlin Plates

Consider a Mindlin plate as shown in Figure 5.1, the relation of displacement \( \mathbf{u} = \begin{bmatrix} w & \theta_x & \theta_y \end{bmatrix}^T \) and strain are discussed by Mindlin [169], having the forms:

\[
\begin{align*}
\chi &= \left\{ \begin{array}{c}
\frac{\partial \theta_x}{\partial x} \\
\frac{\partial \theta_y}{\partial y} \\
\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}
\end{array} \right\} = \begin{bmatrix} 0 & \partial/\partial x & 0 \\
0 & 0 & \partial/\partial y \\
0 & \partial/\partial y & \partial/\partial x \end{bmatrix} \mathbf{u}, \quad (5.1)
\end{align*}
\]

\[
\begin{align*}
\gamma &= \left\{ \begin{array}{c}
\frac{\partial w}{\partial x} + \theta_x \\
\frac{\partial w}{\partial y} + \theta_y
\end{array} \right\} = \begin{bmatrix} \partial/\partial x & 1 & 0 \\
\partial/\partial y & 0 & 1 \end{bmatrix} \mathbf{u}. \quad (5.2)
\end{align*}
\]

The non-linear strain is:
The strain energy, kinetic energy and non-linear strain energy are as follows:

\[
U_e = \frac{1}{2} \int \chi^T D_b \chi dA + \frac{1}{2} \int \gamma^T D_s \gamma dA,
\]  
\[
T = \frac{\rho}{2} \int u^T D_m u dA,
\]  
\[
U_\sigma = \frac{1}{2} \int \theta^T T \theta dA,
\]

where the matrices \(D_b\), \(D_m\) and \(D_s\) are

\[
D_b = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix},
\]

\[
D_m = \begin{bmatrix} t & 0 & 0 \\ 0 & t^3/12 & 0 \\ 0 & 0 & t^3/12 \end{bmatrix},
\]

\[
D_s = kGt \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\]

where \(D = E t^3/12(1-\nu^2)\) is the plate rigidity, \(E\) is Young’s modulus, \(\nu\) is Poisson’s ratio, \(t\) is the thickness of the element, and \(k\) is the shear correction factor and \(G\) is the shear modulus, \(\rho\) is the mass density of the material in Eq. (5.5). The matrix \(T\)
in Eq. (5.6) has the form

\[ T = \begin{bmatrix} T_x & T_{xy} \\ T_y & T_y \end{bmatrix}, \]  

(5.10)

in which \(T_x\) and \(T_y\) are the in-plane compressive forces in \(x\) and \(y\) directions, respectively and \(T_{xy}\) is the in-plane shear force, \(T_x, T_y\) and \(T_{xy}\) are loads per unit length along the plate edge. For plates subject to uniaxial compression on \(x\) direction, \(T_x=1, T_{xy}=T_y=0\), while for plates under isotropic in-plane compression, \(T_x=T_y=1, T_{xy}=0\).

5.3 Fourier \(p\)-element for Mindlin Plates

5.3.1 Shape functions

Fourier \(C^0\) shaped functions \(f_1(\xi) = 1-\xi, f_2(\xi) = \xi, f_i(\xi) = \sin(i-2)\pi\xi\) have been used to solve the axial vibration of a two-node bar [87] and have been extended to the vibration analyses of Mindlin plates [92]. The shape functions are

\[ N_i(\xi, \eta) = f_m(\xi)f_n(\eta), \]  

(5.11)

where

\[ m = 1, \cdots, (p_\xi + 2), n = 1, \cdots, (p_\eta + 2), \quad i = 1, \cdots, (p_\xi + 2)(p_\eta + 2). \]  

(5.12)

and \(p_\xi\) and \(p_\eta\) are the numbers of the trigonometric terms employed in \(\xi\) and \(\eta\).
directions, respectively. The Fourier C⁰ shape functions have zero displacement at each corner node giving additional degrees of freedom (DOFs) along the four edges and in the interior of the element. The DOFs of the four corner nodes are represented by \( m, n \leq 2 \); the DOFs along the four edges are represented if either \( m \) or \( n > 2 \); and the DOFs in the interior are represented if both \( m \) and \( n > 2 \).

The deflection \( w \) and the rotation \( \theta_x, \theta_y \) of the Mindlin plate are interpolated by

\[
\mathbf{u} = \begin{bmatrix} w \\ \theta_x \\ \theta_y \end{bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_i & 0 & 0 & \cdots \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_i & 0 & \cdots \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_i & \cdots \end{bmatrix} \cdot \mathbf{\delta}^\circ \cdot \mathbf{\delta}^r, \quad (5.13)
\]

where \( \mathbf{\delta} \) is the vector of the nodal degree of freedoms of the element.

### 5.3.2 Mapping

The coordinate systems used to define trapezoidal elements are shown in Figure 5.2.

![Diagram](image.png)

**Figure 5.2** The trapezoidal element coordinate transformation

The Jacobian matrix is defined in terms of the Cartesian coordinates at the four corner nodes as
where $e_0 = d_0 - b_0 - a_0$. Then the determinant of Jacobian is $|J| = c_0 \left( a_0 + e_0 \eta \right)$.

### 5.3.3 Stiffness, mass and geometric matrices

The equation of the dynamic stability problems of the Mindlin plates is given by

$$\left( K - \omega^2 M - TG \right) q = 0 ,$$

where $\omega$ and $T$ is the natural frequency and compressive load, respectively, $q$ is the generalized nodal displacements.

The stiffness, mass and geometric stiffness matrices of the element by applying the principle of minimum potential energy can be given as

$$K^e = \int_A B^T_b \mathbf{D}_b \mathbf{B}_b \, dA + \int_A B^T_0 \mathbf{D}_0 \mathbf{B}_0 \, dA$$

$$= \int_0^1 \int_0^1 B^T_b \left( \xi, \eta \right) \cdot \mathbf{D}_b \cdot B_b \left( \xi, \eta \right) \cdot |J| \cdot d\xi d\eta \quad \text{(5.16)}$$

$$+ \int_0^1 \int_0^1 B^T_0 \left( \xi, \eta \right) \cdot \mathbf{D}_0 \cdot B_0 \left( \xi, \eta \right) \cdot |J| \cdot d\xi d\eta ,$$

$$M^e = \int_A \rho \cdot N^T_m \cdot \mathbf{D}_m \cdot \mathbf{N}_d \, dA$$

$$= \rho \int_0^1 \int_0^1 N^T_m \left( \xi, \eta \right) \cdot \mathbf{D}_m \cdot N_m \left( \xi, \eta \right) \cdot |J| \cdot d\xi d\eta , \quad \text{(5.17)}$$
\[ G^e = \int_A C^T T C dA \]
\[ = \int_0^1 \int_0^1 C^T (\xi, \eta) \cdot T \cdot C(\xi, \eta) \cdot |J| \cdot d\xi d\eta, \quad (5.18) \]

where the matrices \( B_b, B_s \) have the forms:

\[
B_b = \begin{bmatrix} 0 & \partial/\partial x & 0 \\ \partial/\partial y & 0 & 0 \\ 0 & \partial/\partial y & \partial/\partial x \end{bmatrix} N, \quad (5.19)
\]

\[
B_s = \begin{bmatrix} \partial/\partial x & 1 & 0 \\ \partial/\partial y & 0 & 1 \end{bmatrix} N, \quad (5.20)
\]

\[
C = \begin{bmatrix} \partial/\partial x & 0 & 0 \\ \partial/\partial y & 0 & 0 \end{bmatrix} N. \quad (5.21)
\]

The coefficients of the stiffness matrix and the mass matrix of Mindlin plates can be found in Zhu [19]. The coefficients of the geometric stiffness matrix are explicitly given in Appendix 6.

### 5.4 Numerical Results

Numerical examples of dynamic stability problems of Mindlin skew and trapezoidal plates subjected to uniaxial loads are considered in Section 5.4.1 and Section 5.4.2. Plates with different shapes and various boundary conditions are investigated in details. Dynamic stability problems of polygonal plates under isotropic in-plane compression are studied in Section 5.4.3. Moreover, the
problems of plate systems composed of plate elements with different thickness are considered in Section 5.4.4.

5.4.1 Square plates and skew plates

Dynamic stability problems of square and skew plates with different boundary conditions are investigated in this section. A skew plate subject to uniaxial compression is shown in Figure 5.3a. Three kinds of boundary conditions with the combinations of free (F), hard type simply supported (S) and clamped (C) are considered (as shown in Figure 5.3b).

For simplicity in our study, we use \( \Omega \) and \( \lambda \) to represent the non-dimensional parameters of frequency \( \omega \) and load \( T \). For a square plate subject to uniaxial compression, the parameters \( \Omega \) and \( \lambda \) are defined as

\[
\Omega = \frac{\omega L_o^2 \sqrt{\rho t/D}}{\pi^2}, \quad \lambda = \frac{T_s L_o^2}{\pi^2 D},
\]

(5.22)
where $T_x$ is the uniaxial compressive loads and $L_b$ is the width of the square plate. For a skew plate, the parameters $\Omega$ and $\lambda$ are defined the same as a square plate, where $L_b$ is the width of the skew plate (see from Figure 5.3a). The relationships of $\Omega$ and $\lambda$ will be shown in the figures below. Convergence study of frequency parameters $\Omega$ for a simply supported square plate has been considered by Leung and Zhu [92], from which it can be seen that the number of the additional trigonometric terms $p_\xi=p_\eta=16$ is enough for the accurate solutions and the number will be chosen to solve the skew and trapezoidal plate problems in Section 5.4.1 and Section 5.4.2. Unless stated otherwise, Poisson’s ratio $\nu$ and the shear correction factor $k$ are chosen as 0.3 and 5/6. The thickness $t$ of the plate is equal to 0.1. For simplicity the trigonometric terms $p_\xi$ is taken the same as $p_\eta$.

Figure 5.4a, Figure 5.5a and Figure 5.6a show the interaction diagrams of SSSS, CCCC and FFFC square plate with $L_b/t=10$, respectively. The corresponding dynamic buckling modes of SSSS, CCCC and FFFC square plate are given in Figure 5.4b-5.6b, respectively. The interaction diagrams of skew plates with $L_a/t=10$, $L_a/L_b=1$, skew angle $\alpha=15^\circ$ are given in Figure 5.7a-5.9a. As shown in Figure 5.4b and Figure 5.5b, the dynamic buckling modes of SSSS plates are very similar to the buckling modes of CCCC plates, then both the buckling modes of SSSS and CCCC square plates can be divided into two types. One type named as type A, is used to describe the case that the square plates buckles at a major axis that the buckling shapes of the plate are generally identical along the axis which is parallel to the square plate edges as shown in Figure 5.4b and 5.5b. Another type defined as type B, is for the plate buckling at another major-axis which is vertical to the major-axis of type A. The frequency and buckling load parameters existed on each line of the interaction diagram has similar buckling modes. To distinguish the lines from each other the mode types and mode numbers of every line are
marked in the interaction diagram (as the symbols $A_1, A_2, A_3, B_1, B_2, B_3$ appeared in the interaction diagrams). The intersect points appearing in the diagram ($B_1$ mode with $A_3$ mode, $B_2$ mode with $A_3$ mode) as shown in Figure 5.4a, indicate energy exchange exists on these points. Since the buckling modes of CCCC plates are very similar to that of SSSS plates, the buckling modes of CCCC skew plates and CCCC trapezoidal plates will not be provided in this section. It is found in Figure 5.6b that the buckling modes of FFFC square plate can be separated into three types: type A (horizontal major-axis), type B (vertical major-axis) and type C (no major-axis). Moreover, as shown in Figure 5.7b, the buckling modes of skew plates are very similar to square plates, besides of the major-axis rotating an angle $(\pi/2-\alpha)/2$, where $\alpha$ is the skew angle of the skew plate as shown in Figure 5.3a.

![Figure 5.4a](image-url)  

*Figure 5.4a A square plate under uniaxial compression with SSSS boundary condition*
Chapter 5: Dynamic stability of Mindlin plates

Figure 5.4b Dynamic buckling modes of a square plate under uniaxial compression with SSSS boundary condition

Figure 5.5a A square plate under uniaxial compression with CCCC boundary condition
Figure 5.5b Dynamic buckling modes of a square plate under uniaxial compression with CCCC boundary condition

Figure 5.6a A square plate under uniaxial compression with FFFC boundary condition
Figure 5.6b Dynamic buckling modes of a square plate under uniaxial compression with FFFC boundary condition

Figure 5.7a A skew plate under uniaxial compression with SSSS boundary condition
Figure 5.7b Dynamic buckling modes of a skew plate under uniaxial compression with SSSS boundary condition.

Figure 5.8 A skew plate under uniaxial compression with CCCC boundary condition.
Dynamic stability problems of symmetrical trapezoidal plate subject to uniaxial compression on two unparallel sides are considered in this section. Figure 5.10a shows the dimension of the trapezoidal plate, where $L_c=3/5$, $L_b=1$, $L_b/t=10$, $\beta=15^\circ$. For a trapezoidal plate, the frequency parameter and buckling load factor can be defined in Eq. (5.15), where $L_b$ is the height of the trapezoidal plate. Interaction diagrams of trapezoidal plates with SSSS and CCCC boundary conditions (as shown in Figure 5.10b) are given in Figure 5.11a and Figure 5.12, respectively. Figure 5.11b shows the buckling modes of SSSS trapezoidal plates, which are also similar to buckling modes of square plates and skew plates.
Chapter 5: Dynamic stability of Mindlin plates

Figure 5.10a A trapezoidal plate subject uniaxial compression

Figure 5.10b A trapezoidal plate with SSSS and CCCC boundary conditions

Figure 5.11a A trapezoidal plate under uniaxial compression with SSSS boundary condition
Figure 5.11b Dynamic buckling modes of a trapezoidal plate under uniaxial compression with SSSS boundary condition

Figure 5.12 A trapezoidal plate under uniaxial compression with CCCC boundary condition

5.4.3 Polygonal plates
An equilateral triangular, square, hexagonal and octagonal thick plates with length vs. thickness ratio $L_p/t=10$ subject to uniform in-plane compressive loads are
shown in Figure 5.13. The polygons are discretized as combinations of trapezoidal elements and rectangular elements around the shape center of the polygons. Two boundary conditions of full simply supported and full clamped on all sides of the plate are considered. The frequency parameter $\Omega$ of CFF equilateral triangular thick plate with $L_p/t=10$, $k=\pi^2/12$ are given in Table 5.1. It can be seen that $p_\xi=p_\eta=8$ are in excellent agreement with the Ritz Method [179]. The buckling load factors for CCC and CFF equilateral triangular plates ($L_p/t=10$, $k=0.823$, $p_\xi=p_\eta=8$) subject to isotropic in-plane compression are 10.1395 and 0.6695, respectively, which are agreed well with the results of Xiang [180] (10.1189 and 0.6684). The trigonometric terms $p_\xi=p_\eta=8$ will be taken in the computations of triangular, square and hexagonal plates. Since the octagonal plate are divided into 6 elements, which is larger than the number 4 of hexagonal plates, to save the computational time, $p_\xi=p_\eta=7$ will be used to solve problems of octagonal plates.

The frequency parameters $\Omega$ and buckling load intensity factors $\lambda$ for triangular plates, square plates, hexagon plates and octagon plates are listed in Table 5.2-5.5, respectively. In this section the parameters $\Omega$ and $\lambda$ are defined in Eq. (5.15), where the parameter $L_b$ should be replaced by side length $L_p$ of polygonal plates and $T_s$ should be changed to the isotropic loads $T$. By comparing Table 5.2-5.3 with Table 5.4-5.5, it is found that the multiple eigenvalues (including frequency parameters and buckling loads factors) of triangular and square plates are obtained as equally as expected, while for hexagonal and octagonal plates, the expected multiple eigenvalues become a little different with each other. It is possibly due to the elements composed of the hexagonal and octagonal plates that are not strictly centrally symmetric about the shape center of the polygons. The eigenvalues that are not as equally as expected are listed as follows: the frequencies of CCCCCC hexagon plates (6.7749 and 6.7766), the buckling loads of CCCCCC hexagon
plates (7.6618 and 7.6637); the frequencies of SSSSSSSSS octagon plates (5.2711 and 5.2811, 8.8614 and 8.8626), the frequencies of CCCCCCCC octagon plates (6.9546 and 6.9549), the buckling loads of SSSSSSSSS octagon plates (5.0473 and 5.0652, 8.0692 and 8.0717), the buckling loads of CCCCCCCC octagon plates (7.8329 and 7.8376).

The interaction diagrams of the four kinds of polygonal plates with fully simply supported boundary conditions are given in Figures 5.14, 5.16, 5.18 and 5.20, respectively. Interaction diagrams for plates under fully clamped boundary conditions are shown in Figures 5.15, 5.17, 5.19 and 5.21, respectively. It can be found from the interaction diagrams that the points the contour lines intersect with \( x \) or \( y \) axis are corresponding to the frequency parameters or buckling load intensity factors as listed in Table 5.2-5.5, respectively. However, the multiple value buckling modes of triangular and square plates are disappeared in the interaction diagrams. On the contrary, since the elements composed of the hexagonal plates (four trapezoids ) and octagonal plates (four trapezoids and two rectangles) that are not symmetric around the shape center, each expected multiple value mode are separated into two nearby modes, and finally appeared in the interaction diagrams. The eigenvalues disappeared in the interaction diagrams are underlined in Table 5.2-5.5.
Hexagon

Octagon

Figure 5.13 Polygonal plates subject to isotropic in-plane compression

Table 5.1 Natural frequency parameters $\Omega = \omega L_p^2 \sqrt{\rho t / D} (L_p/t=10, \ k=\pi^2/12, \ \nu=0.3)$ for a CFF equilateral triangular plate

<table>
<thead>
<tr>
<th>Mode number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present ($p_\xi=p_\eta=6$)</td>
<td>8.6586</td>
<td>31.4907</td>
<td>34.8882</td>
<td>75.7505</td>
<td>76.4565</td>
</tr>
<tr>
<td>Present ($p_\xi=p_\eta=7$)</td>
<td>8.6526</td>
<td>31.4727</td>
<td>34.8717</td>
<td>75.6202</td>
<td>76.3261</td>
</tr>
<tr>
<td>Present ($p_\xi=p_\eta=8$)</td>
<td>8.6510</td>
<td>31.4484</td>
<td>34.8453</td>
<td>75.5706</td>
<td>76.2834</td>
</tr>
<tr>
<td>Reference [179]</td>
<td>8.646</td>
<td>31.41</td>
<td>34.81</td>
<td>75.4</td>
<td>76.15</td>
</tr>
</tbody>
</table>

Table 5.2 Natural frequency parameters $\Omega = \omega L_p^2 \sqrt{\rho t / D} / \pi^2$ and buckling load intensity factors $\lambda = TL_p^2 / (\pi^2 D)$ for a SSS and CCC equilateral triangular plate ($L_p/t=10, \ k=5/6, \ \nu=0.3$)

<table>
<thead>
<tr>
<th>Boundary conditions</th>
<th>SSS</th>
<th>CCC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mode number</td>
<td>Frequencies</td>
<td>Buckling loads</td>
</tr>
<tr>
<td>1</td>
<td>4.8937</td>
<td>4.6394</td>
</tr>
<tr>
<td>2</td>
<td>10.4184</td>
<td>9.2189</td>
</tr>
<tr>
<td>3</td>
<td>10.4184</td>
<td>9.2189</td>
</tr>
<tr>
<td>4</td>
<td>-</td>
<td>13.3368</td>
</tr>
<tr>
<td>5</td>
<td>-</td>
<td>14.0043</td>
</tr>
<tr>
<td>6</td>
<td>-</td>
<td>14.0043</td>
</tr>
</tbody>
</table>
Table 5.3 Natural frequency parameters $\Omega = \omega L_p^2 \sqrt{\rho t / D / \pi^2}$ and buckling load intensity factors $\lambda = TL_p^2 / (\pi^2 D)$ for a SSSS and CCCC square plate ($L_p/t=10$, $k=5/6$, $\nu=0.3$)

<table>
<thead>
<tr>
<th>Mode number</th>
<th>Frequencies</th>
<th>Buckling loads</th>
<th>Frequencies</th>
<th>Buckling loads</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.9326</td>
<td>1.8950</td>
<td>3.3016</td>
<td>4.5600</td>
</tr>
<tr>
<td>2</td>
<td>4.6091</td>
<td>4.3824</td>
<td>6.2963</td>
<td>7.2211</td>
</tr>
<tr>
<td>3</td>
<td>4.6091</td>
<td>4.3824</td>
<td>6.2963</td>
<td>7.2211</td>
</tr>
<tr>
<td>4</td>
<td>7.0722</td>
<td>6.5285</td>
<td>8.8229</td>
<td>9.2279</td>
</tr>
<tr>
<td>5</td>
<td>8.6269</td>
<td>7.8161</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>6</td>
<td>8.6269</td>
<td>7.8161</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>7</td>
<td>-</td>
<td>9.5243</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>8</td>
<td>-</td>
<td>9.5243</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 5.4 Natural frequency parameters $\Omega = \omega L_p^2 \sqrt{\rho t / D / \pi^2}$ and buckling load intensity factors $\lambda = TL_p^2 / (\pi^2 D)$ for a SSSSSS and CCCCCC hexagonal plate ($L_p/t=10$, $k=5/6$, $\nu=0.3$)

<table>
<thead>
<tr>
<th>Mode number</th>
<th>Frequencies</th>
<th>Buckling loads</th>
<th>Frequencies</th>
<th>Buckling loads</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.8688</td>
<td>2.0002</td>
<td>3.5162</td>
<td>4.8074</td>
</tr>
<tr>
<td>2</td>
<td>3.8925</td>
<td>3.4443</td>
<td>6.7749</td>
<td>7.6618</td>
</tr>
<tr>
<td>3</td>
<td>4.0490</td>
<td>4.6107</td>
<td>6.7766</td>
<td>7.6637</td>
</tr>
<tr>
<td>4</td>
<td>5.9030</td>
<td>4.7810</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>5</td>
<td>-</td>
<td>5.8282</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>
Table 5.5 Natural frequency parameters $\Omega = \omega L^2 \sqrt{\rho t / D} / \pi^2$ and buckling load intensity factors $\lambda = TL^2 / (\pi^2 D)$ for a SSSSSSSS and CCCCCCCC octagonal plate ($L_p/t=10$, $k=5/6$, $\nu=0.3$)

<table>
<thead>
<tr>
<th>Mode number</th>
<th>Frequency SSSSSSSS</th>
<th>Frequency CCCCCCCC</th>
<th>Buckling load SSSSSSSS</th>
<th>Buckling load CCCCCCCC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.2428</td>
<td>3.6090</td>
<td>2.2769</td>
<td>4.9216</td>
</tr>
<tr>
<td>2</td>
<td>5.2711</td>
<td>6.9564</td>
<td>5.0473</td>
<td>7.8329</td>
</tr>
<tr>
<td>3</td>
<td>5.2811</td>
<td>6.9546</td>
<td>5.0652</td>
<td>7.8376</td>
</tr>
<tr>
<td>4</td>
<td>8.8614</td>
<td>-</td>
<td>8.0692</td>
<td>-</td>
</tr>
<tr>
<td>5</td>
<td>8.8626</td>
<td>-</td>
<td>8.0717</td>
<td>-</td>
</tr>
</tbody>
</table>

Figure 5.14 A triangular plate under isotropic compression with SSS boundary condition
Figure 5.15 A triangular plate under isotropic compression with CCC boundary condition

Figure 5.16 A square plate under isotropic compression with SSSS boundary condition
Figure 5.17 A square plate under isotropic compression with CCCC boundary condition

Figure 5.18 A hexagonal plate under isotropic compression with SSSSSS boundary condition
Figure 5.19 A hexagonal plate under isotropic compression with CCCCCC boundary condition

Figure 5.20 A octagonal plate under isotropic compression with SSSSSSSSSS boundary condition
5.4.4 Plate systems with different thicknesses plate elements

5.4.4.1 Systems of two elements
Problems of two square plate systems composed of two rectangular (case 1) or two trapezoidal plate elements (case 2) are shown in Figure 5.22a and Figure 5.22b. The dimensions of the plate elements are chosen as \( L_p = 1 \), \( t_2 = 0.1 \) and the thickness \( t_1 \) is changed from 0.1 to 0.2. The frequency and buckling load changing with the variation of the thickness \( t_1 \) of the plate element are tabulated in Table 5.6 and shown in Figure 5.23. Interaction diagrams of these two plate systems subject to uniaxial compression in the \( x \) direction with FFFC boundary conditions are given in Figures 5.24-5.25. The parameters \( L_p \) and \( t \) in \( \Omega \) and \( \lambda \) are replaced by the width of the plate system \( L_p \) and the thickness of the thinner plate element \( t_2 \). The trigonometric terms \( p_\zeta = p_\eta = 10 \) are chosen.
It can be seen from Table 5.6 and Figure 5.23 that the frequency of the square plate composed of two rectangular plates (case 1) are a little higher than that of plate composed of two trapezoidal plates (case 2). On the contrary, the buckling loads for case 1 are slightly less than that of case 2. When the thickness $t_1$ varied from 0.1 to 0.2, the frequencies and buckling loads increased almost linearly with the thicknesses of the thicker plate elements $t_1$. As shown in Figure 5.24 and 5.25, the interaction diagrams for the two cases when $t_1=0.12$ are very similar to that of uniform thickness square plates given in Figure 5.6a.

Figure 5.22a Square plate systems with two rectangular plates

Figure 5.22b Square plate systems with two trapezoidal plates
Table 5.6 Natural frequency parameters $\Omega = \omega t_p^2 \sqrt{\rho t_1/D}/\pi^2$ and buckling load intensity factors $\lambda = T_c t_p^3/(\pi^2 D)$ for a FFFC square plate composed of two plate elements with different thickness $t_1$ ($L_p/t_2=10$, $k=5/6$, $\nu=0.3$)

<table>
<thead>
<tr>
<th>$t_1/L_p$</th>
<th>Frequencies</th>
<th>Buckling loads</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Two Rectangles</td>
<td>Two Trapezoids</td>
</tr>
<tr>
<td>0.10</td>
<td>0.3480</td>
<td>0.3480</td>
</tr>
<tr>
<td>0.12</td>
<td>0.4448</td>
<td>0.4440</td>
</tr>
<tr>
<td>0.14</td>
<td>0.5420</td>
<td>0.5403</td>
</tr>
<tr>
<td>0.16</td>
<td>0.6366</td>
<td>0.6341</td>
</tr>
<tr>
<td>0.18</td>
<td>0.7262</td>
<td>0.7231</td>
</tr>
<tr>
<td>0.20</td>
<td>0.8090</td>
<td>0.8053</td>
</tr>
</tbody>
</table>

Figure 5.23 Natural frequency and buckling loads with respect to the thickness $t_1$ for a square plate composed of two rectangular plates or two trapezoidal plates.
Figure 5.24 Interaction diagram of a plate system composed of two rectangular elements with FFFC boundary conditions ($t_1=0.12$, $t_2=0.1$, $L_p=1$)

Figure 5.25 Interaction diagram of a plate system composed of two trapezoidal elements with FFFC boundary conditions ($t_1=0.12$, $t_2=0.1$, $L_p=1$)
5.4.4.2 Systems of four elements

Two square plate systems made up of four rectangular (case 1) or trapezoidal plate elements (case 2) subject to isotropic in-plane compression are considered in this section. The dimensions of the two systems are shown in Figure 5.26. The dimensions of the four plate elements are taken as \( t_2=t_4=0.1, L_p=1 \) and \( t_1=t_3 \) varied from 0.1 to 0.2. Similar to the systems composed of two elements, the parameters \( L_b \) and \( t \) in \( \Omega \) and \( \lambda \) are instead of width of the plate system \( L_p \) and thickness of the thinner plate element \( t_2 \) or \( t_4 \). The trigonometric terms \( p_\xi=p_\eta=8 \) are used to achieve the results.

The frequency and buckling load varying with the thickness \( t_1 \) of the plate are shown in Table 5.7 and Figure 5.27. It is observed that the frequencies and buckling loads of CCCC square plate systems for case 1 are both smaller than that of case 2. The frequencies and buckling loads increase linearly with the thickness \( t_1=t_3 \). The interaction diagrams of the two plate systems with thickness \( (t_1=t_3=0.12, t_2=t_4=0.1, L_p=1) \) are given in Figures 5.28 and 5.29, which show similar phenomenon to square plates with uniform thickness that was given in Figure 5.17.

![Figure 5.26 Dimensions of plate systems composed of four plate elements](image)
Table 5.7 Natural frequency parameters \( \Omega = \omega L_p^2 \sqrt{\rho t D / \pi^2} \) and buckling load intensity factors \( \lambda = TL_p^2 / (\pi^2 D) \) for a CCCC square plate system composed of four plate elements with different thickness \( t_1 \) \((L_p/t_2=10, k=5/6, \nu=0.3)\)

<table>
<thead>
<tr>
<th>( t_1/L_p )</th>
<th>Four Rectangles</th>
<th>Four Trapezoids</th>
<th>Four Rectangles</th>
<th>Four Trapezoids</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>3.3016</td>
<td>3.3016</td>
<td>4.5600</td>
<td>4.5603</td>
</tr>
<tr>
<td>0.12</td>
<td>3.5590</td>
<td>3.6004</td>
<td>5.7862</td>
<td>5.9845</td>
</tr>
<tr>
<td>0.14</td>
<td>3.7953</td>
<td>3.8722</td>
<td>7.0018</td>
<td>7.4223</td>
</tr>
<tr>
<td>0.16</td>
<td>4.0173</td>
<td>4.1238</td>
<td>8.2363</td>
<td>8.8834</td>
</tr>
<tr>
<td>0.18</td>
<td>4.2292</td>
<td>4.3600</td>
<td>9.4867</td>
<td>10.3519</td>
</tr>
<tr>
<td>0.20</td>
<td>4.4327</td>
<td>4.5831</td>
<td>10.6839</td>
<td>11.7279</td>
</tr>
</tbody>
</table>

Figure 5.27 Natural frequency and buckling loads with respect to the thickness \( t_1 \) for a square plate composed of four rectangular plates or four trapezoidal plates.
Figure 5.28 Plate system composed of four rectangular elements with CCCC boundary conditions ($t_1=t_3=0.12$, $t_2=t_4=0.1$, $L_p=1$)

Figure 5.29 Plate system composed of four trapezoidal elements with CCCC boundary conditions ($t_1=t_3=0.12$, $t_2=t_4=0.1$, $L_p=1$)
5.5 Conclusion

An analytically integrated trapezoidal Fourier $p$-element is extended to solve dynamic stability of Mindlin plates and plate systems. Different shapes and various boundary conditions are considered. The dynamic buckling modes of square plates, skew plates and trapezoidal plates are studied. The natural frequency and buckling load intensity factors of polygonal plates and plate systems are obtained. The influence of the thickness of the composed plate elements on the frequencies and buckling loads of the plate systems is studied in detail.
CHAPTER 6

POLYNOMIAL \( p \)-ELEMENTS FOR VIBRATION, BUCKLING OF OPEN, THIN, SHELL PANELS

The use of polynomial \( p \)-elements as shape functions will converge much faster than the use of Fourier series when the beams or plates are of \( C^0 \) continuity. This chapter extends the polynomial \( C^0 \) \( p \)-elements to study the dynamic stability of open thin shell panels, including cylindrical shell panels, truncated conical shell panels and spherical shell panels. By using polynomial \( p \)-elements the stiffness, mass and geometric stiffness matrices of the shell panels can be integrated analytically. The natural frequency parameters of cylindrical, conical and spherical shell panels are obtained and compared with the existed solutions. The buckling problems of axially compressed cylindrical shell panels are considered and the results are compared with those of ANSYS. The stresses in the axially compressed cylindrical shell panels are assumed to be uniform. Since the stress distribution in the conical and spherical shell panels is rather complicated, it is hard to obtain analytical matrices using polynomial \( p \)-elements. Therefore, the buckling problems of conical and spherical shell problems will not be studied in this chapter.
6.1 Introduction

Thin shell structures have a wide range of engineering applications, particularly in aerospace, cooling towers, marine and pressure vessels. For analysis and design, the free vibration and buckling capacity of the shell panels is an important item of consideration. A large amount of research is existed in vibration problems of open shell panels, especially on cylindrical shell panels and conical shell panels. The early vibration analysis of open cylindrical shell panels is given by Lessia [181-183]. A \( pb \)-2 Ritz formulation was introduced by Lim and Liew [184] to solve flexural vibration of shallow cylindrical shells of rectangular planform. Three-dimensional vibration of cylindrical shell panels were considered by Lim [185] using a three-dimensional elasticity approach and later by Liew and Bergman [186] using continuum and discrete approaches. The \( h-p \) version of the finite element method is used by Bardell [45] to solve thin completely free open cylindrical shell panels. Non-linear vibration of thick cylindrical shallow shell panels was studied by Ribeiro [68] using hierarchical finite element. Moreover, open cylindrical shells with stepped thickness variations are investigated by Zhang and Xiang [187].

The early study on vibration analysis of open conical shell panels was given by Srinivasan and Krishnan [188] using Donnell’s shell theory and an integral equation technique. A spline finite strip method was developed by Cheung et al [189] to study singly curved shells. Lim and his co-authors considered vibration problems of shallow conical shells using a Rayleigh-Ritz method in a conical co-ordinate system [190], and an extension of the paper presented by Lim et al [190] was given in Lim and Kitipornchai [191]. After having the investigation on open cylindrical shell panels [45], Bardell extended the \( h-p \) finite element to study
the vibration of open thin isotropic shell panels [48] and open conical sandwich panels and conical sandwich frusta [49]. Recent works on the vibration problems of conical shell panels were considered by Lam and Li [192] using a generalized differential quadrature method, by Zhao et al [193] using the element-free $kp$-Ritz method and by Civalek [194] using the method of discrete singular convolution.

Studies on spherical shell panels are few when compared with cylindrical shell panels and conical shell panels. By using spline finite element Fan and Luah [195] solved some thin shell structures, containing spherical panels, cylindrically curved trapezoidal fan blade and hyperboloidal shells. Large amplitude vibration behavior of square cylindrical shell panels and spherical shell panels with simply-supported edges are considered by Shin [196] and Liew and Peng [197] using the Ritz method.

Buckling problems of shells panels are limited to open cylindrical panels and spherical panels. Early works on cylindrical panels with a free edge are presented by Chu et al [198-199], Yang and Guralnick [200]. Matusunaga [201] considered the vibration and stability of thick simply supported shallow shells subject to in-plane stresses, including spherical, cylindrical, and hyperbolic paraboloidal shell panels. Magnucki and Mackiewicz [202] studied the elastic buckling of an axially compressed cylindrical panel with three edges simply supported and one edge free.

The hierarchical finite element method ($p$-method) is a useful method applicable to vibration and buckling problems of plates and shells. Vibration problems of thin cylindrical shell panels are considered by Bardell [45] using K-orthogonal polynomials satisfying both displacement and slope continuity ($C^1$ continuity).
Geometrically non-linear vibration of cylindrical thick isotropic shallow shell panels was studied by Ribeiro [68] using Legendre orthogonal polynomials satisfying displacement continuity (C^0 continuity). The analytically integration of the stiffness matrices and mass matrices of cylindrical shell panels are obtained by both of Bardell [45] and Ribeiro [68]. Meanwhile, Bardell [48] using an h-p version of the finite element method to solve free vibration of thin conical shell panels. The complexity of the theory of conical shell panels does not allow the stiffness and mass matrices to be integrated explicitly, a Gauss-Legendre numerical integration method was used instead by Bardell [48]. Therefore, in our study, Legendre orthogonal polynomial series satisfying C^0 continuity are used as shape functions, so that the stiffness, mass and geometric stiffness matrices of the shell panels can be analytically integrated by MATLAB®. Meanwhile, the natural frequencies of open cylindrical, conical and spherical shell panels are considered and compared with the existed solutions. Since the buckling problems of open shell panels subject to axial compression are seldom considered, the present buckling loads will be compared with the results of finite element program ANSYS (h-version) [168]. The effects of aspect ratio, circumferential angles of the shell panels on the frequencies and buckling loads of the shell panels will be considered in details.

6.2 Governing Equations

Considering an open, thin shell panel as shown in Figure 6.1, having the following assumptions:
Assumptions:
The distance $\zeta$ away from the mid-surface is assumed to be much smaller than the curvature radius $R$, i.e. $1-\zeta/R \approx 1$. Therefore, the differential $dV$ and $dS$ of the shell panels can be expressed as

\[
dV = AB \left(1 - \frac{\zeta}{R_\alpha}\right) \left(1 - \frac{\zeta}{R_\beta}\right) d\alpha d\beta d\zeta \approx AB d\alpha d\beta d\zeta, \\
dS = \sqrt{m^2 \left(1 - \frac{\zeta}{R_\alpha}\right)^2 + l^2 \left(1 - \frac{\zeta}{R_\beta}\right)^2} \, ds d\zeta \approx ds d\zeta, \tag{6.1}
\]

where $l$, $m$ are direction cosines of the outward boundary normal, and the geometry of the shell panel is given in Figure 6.1. The assumption is consistent with shallow shells.

Geometry:

![Figure 6.1 Geometry of a thin shell panel](image)

The unit vectors along $(\alpha, \beta, \zeta)$ before and after deformation are respectively given by
where a subscript comma denotes partial differentiation. Superscript ‘0’ denotes un-deformed state and subscript ‘0’ denotes mid-surface. Subscript ‘0’ may be deleted when no confusion arrives. Point \( P_0^0 \) at the un-deformed mid-surface is denoted by the position vector \( r_0^0 = r_0^0(\alpha, \beta) \) which is a function of the coordinates \((\alpha, \beta)\). The unit vectors can also be written as

\[
\begin{align*}
a^0 = \frac{\partial r_0^0}{A \partial \alpha}, & \quad b^0 = \frac{\partial r_0^0}{B \partial \beta}, & \quad A = \left| \frac{\partial r_0^0}{\partial \alpha} \right|, & \quad B = \left| \frac{\partial r_0^0}{\partial \beta} \right|,
\end{align*}
\]

so that \( n^0 = a^0 \times b^0 \). Along the coordinate lines, displacements having distance \( \varsigma \) away from the mid-surface are related to the displacements \((u,v,w)\) and rotation \((u_1,v_1)\) at the mid-surface by

\[
\mathbf{u} = \begin{pmatrix} u + \varsigma u_1 \\ v + \varsigma v_1 \\ w \end{pmatrix}, \quad \mathbf{r} = r_0^0 + (u + \varsigma u_1) a^0 + (v + \varsigma v_1) b^0 + w n^0, \quad (6.4)
\]

where \( \mathbf{r}, r_0^0 \) are position vectors of points at distance \( \varsigma \) away from the mid-surface during deformation and before deformation respectively.

The primary unknowns in the formulation are the generalized displacements \((u,v,w,u_1,v_1)\). By differentiations, one has
Chapter 6: Vibration and buckling of open thin shell panels

\[
\begin{bmatrix}
\frac{\partial}{\partial \alpha} a^0 \\
\frac{\partial}{\partial \beta} b^0 \\
\frac{\partial}{\partial \zeta} n^0
\end{bmatrix}
= \begin{bmatrix}
0 & -\frac{A_\beta}{B} & \frac{A}{R_\alpha} \\
\frac{A_\beta}{B} & 0 & 0 \\
-\frac{A}{R_\alpha} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
a^0 \\
b^0 \\
n^0
\end{bmatrix},
\quad \begin{bmatrix}
0 & B_\alpha & 0 \\
-\frac{B_\alpha}{A} & 0 & \frac{B}{R_\beta} \\
0 & -\frac{B}{R_\beta} & 0
\end{bmatrix}
\begin{bmatrix}
a^0 \\
b^0 \\
n^0
\end{bmatrix},
\quad (6.5)
\]

Linear Strains:

The linear strains can be obtained as Eq. (6.7):

\[
\begin{bmatrix}
\varepsilon_\alpha \\
\varepsilon_\beta \\
\gamma_{\alpha\beta} \\
\gamma_{\alpha\zeta} \\
\gamma_{\beta\zeta}
\end{bmatrix} = \begin{bmatrix}
\varepsilon_{\alpha 0} - \zeta \kappa_\alpha \\
\varepsilon_{\beta 0} - \zeta \kappa_\beta \\
\gamma_{\alpha 0} - 2\zeta \kappa_{\alpha\beta} \\
\gamma_{\alpha \zeta} \\
\gamma_{\beta \zeta}
\end{bmatrix},
\quad (6.7)
\]

and \( \varepsilon^0 \) are the generalized strains at mid-surface which are denoted as \( \varepsilon^0 = \{ \varepsilon_\alpha, \varepsilon_\beta, \gamma_{\alpha\beta}, \gamma_{\alpha\zeta}, \gamma_{\beta\zeta}, \kappa_\alpha, \kappa_\beta, \kappa_{\alpha\beta} \} \) for subsequent study.

Generalized linear stress and strain at mid-surface:

The generalized linear stress vector is given by

\[
\sigma^0 = \{ N_\alpha, N_\beta, N_{\alpha\beta}, Q_\alpha, Q_\beta, M_\alpha, M_\beta, M_{\alpha\beta} \},
\quad (6.8)
\]
Generalized strains:

The components of the generalized strain are given by

\[ \varepsilon_a = l_{11}, \varepsilon_{\beta} = l_{22}, \gamma_{\alpha\beta} = l_{12} + l_{21}, \kappa_\alpha = -m_{11}, \kappa_\beta = -m_{22}, \]

\[ \gamma_{\alpha \zeta} = u_1 + l_{31}, \gamma_{\beta \zeta} = v_1 + l_{32}, 2 \kappa_{\alpha \beta} = -m_{21} - m_{12} + \frac{l_{12}}{R_\alpha} + \frac{l_{21}}{R_\beta}. \]

(6.9)

In terms of the generalized displacements,

\[ \begin{bmatrix} \varepsilon_a \\ \varepsilon_{\beta} \\ \gamma_{\alpha\beta} \\ \gamma_{\alpha \zeta} \\ \gamma_{\beta \zeta} \\ 2 \kappa_{\alpha \beta} \end{bmatrix} = \begin{bmatrix} l_{11} \\ l_{22} \\ l_{12} + l_{21} \\ u_1 + l_{11} \\ v_1 + l_{21} \\ -m_{11} \\ -m_{22} \end{bmatrix} \begin{bmatrix} \frac{\partial a}{A} \\ \frac{A_{\beta}}{AB} \\ \frac{1}{R_\alpha} \\ \frac{B_{\alpha}}{R_\beta} \\ \frac{1}{AB} \\ \frac{1}{R_\alpha} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ u_t \\ v_t \end{bmatrix} = \begin{bmatrix} \frac{\partial a}{A} - \frac{A_{\beta}}{AB} \\ \frac{A_{\beta}}{AB} - \frac{\partial a}{A} \\ \frac{1}{R_\alpha} \\ \frac{B_{\alpha}}{R_\beta} \\ \frac{1}{AB} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ u_t \\ v_t \end{bmatrix} + \begin{bmatrix} \frac{1}{R_\alpha} \\ \frac{1}{R_\beta} \\ \frac{1}{AB} \end{bmatrix} \begin{bmatrix} \frac{\partial a}{A} \\ \frac{A_{\beta}}{AB} \\ \frac{1}{R_\alpha} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ u_t \\ v_t \end{bmatrix} \]

(6.10)

Nonlinear strains (with linear curvatures):

When including the second order effects with linearized curvatures, the strain components are separated into the linear parts and the nonlinear parts,
\[
\begin{align*}
\begin{bmatrix}
\varepsilon_a \\
\varepsilon_\beta \\
\gamma_{a\beta} \\
\gamma_{a\varepsilon} \\
\gamma_{\beta\varepsilon} \\
\kappa_a \\
\kappa_\beta \\
2\kappa_{a\beta}
\end{bmatrix} &= \begin{bmatrix}
\frac{1}{2} \left( (1 + l_{11})^2 + L_{21}^2 + L_{31}^2 - 1 \right) \\
\frac{1}{2} \left( L_{12}^2 + (1 + l_{22})^2 + L_{32}^2 - 1 \right) \\
(1 + l_{11})l_{12} + (1 + l_{22})l_{21} + l_{31}l_{32} \\
& \quad u_i + l_{31} \\
& \quad v_i + l_{32} \\
& \quad -m_{11} \\
& \quad -m_{22} \\
& \quad -m_{21} - m_{12} + \frac{l_{12}}{R_\alpha} + \frac{l_{21}}{R_\beta}
\end{bmatrix} = \varepsilon^0 + \varepsilon', \quad (6.11)
\end{align*}
\]

where \( \varepsilon^0 \) is the linear strain and only the first 3 components of \( \varepsilon' \) are non-zero.

We may just write

\[
\varepsilon' = \frac{1}{2} \begin{bmatrix}
l_{11}^2 + l_{21}^2 + l_{31}^2 \\
l_{12}^2 + l_{22}^2 + l_{32}^2 \\
2(l_{11}l_{12} + l_{22}l_{21} + l_{31}l_{32})
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
l_{11} & l_{21} & l_{31} & 0 & 0 & 0 \\
l_{12} & l_{22} & l_{32} & l_{11} & l_{21} & l_{31} \\
l_{12} & l_{22} & l_{32} & l_{11} & l_{21} & l_{31}
\end{bmatrix} = \frac{1}{2} \mathbf{H}_0, \quad (6.12)
\]

\[
\delta \varepsilon' = \begin{bmatrix}
l_1 d l_{11} + l_2 d l_{21} + l_3 d l_{31} \\
l_1 d l_{12} + l_2 d l_{22} + l_3 d l_{32} \\
l_1 d l_{12} + l_3 d l_{21} + l_1 d l_{11} + l_2 d l_{23} + l_3 d l_{31}
\end{bmatrix} = \begin{bmatrix}
l_1 & l_2 & l_3 & 0 & 0 & 0 \\
l_1 & l_2 & l_3 & l_1 & l_2 & l_3 \\
l_1 & l_2 & l_3 & l_1 & l_2 & l_3
\end{bmatrix} d \mathbf{H}_0 = (d \mathbf{H}) \mathbf{\theta}. \quad (6.13)
\]

For any vector \( \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \), one has,
Chapter 6: Vibration and buckling of open thin shell panels

\[
\mathbf{H}^T \mathbf{x} = \begin{bmatrix}
I_{11} & 0 & I_{12} \\
I_{21} & 0 & I_{22} \\
I_{31} & 0 & I_{32}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
x_1 \mathbf{I} & x_2 \mathbf{I} & x_3 \mathbf{I}
\end{bmatrix}
\begin{bmatrix}
I_{11} \\
I_{21} \\
I_{31} \\
0 \\
I_{12} \\
I_{22} \\
I_{32}
\end{bmatrix}
= \mathbf{X} \theta, \quad (6.14)
\]

where \( \mathbf{I} \) is a 3-identity matrix.

If \( \mathbf{x} = \left\{ N_\alpha, N_\beta, N_{\alpha\beta} \right\} \) are the self-equilibrating initial generalized stress, then

\[
\mathbf{H}^T \mathbf{x} = \mathbf{X} \theta = \begin{bmatrix}
N_\alpha \mathbf{I} & N_{\alpha\beta} \mathbf{I} \\
N_{\alpha\beta} \mathbf{I} & N_\beta \mathbf{I}
\end{bmatrix} \theta. \quad (6.15)
\]

Energy:

The linear strain energy is given by

\[
U = \frac{1}{2} \int \mathbf{e}^0 : \mathbf{D}_k : \mathbf{e}^0 \cdot A \cdot B \, d\alpha d\beta, \quad (6.16)
\]

where \( \mathbf{D}_k \) is the matrix elasticity constants. For isotropic materials,
where the in-plane rigidity $C = E\!/\!(1-\nu^2)$, bending rigidity $D = E\!/\![12(1-\nu^2)]$, shear modulus $G = E/[2(1+\nu)]$, and shear factor $k = 5/6$ for isotropic materials, $E$ and $G$ are Young’s modulus and shear modulus of the shells, $\nu$ is Poisson’s ratio, and $t$ is the thickness of the shell panels.

The kinetic energy magnitude is

$$T = \frac{1}{2} \int \int \mathbf{u}^T \mathbf{D}_m \mathbf{u} \cdot \mathbf{A} \cdot \mathbf{B} \cdot d\alpha d\beta,$$

where

$$\mathbf{D}_m = \begin{bmatrix} \rho t & \rho t & \rho t^3/12 \\ \rho t & \rho t & \rho t^3/12 \\ \rho t^3/12 & \rho t^3/12 & \rho t^3/12 \end{bmatrix},$$

and $\rho$ is the mass density of the shell panels.
The strain energy in buckling analysis is given by

\[ U_N = \frac{1}{2} \int_0^\theta \mathbf{0}^T \mathbf{X} \cdot \mathbf{A} \cdot \mathbf{B} \cdot \mathbf{d} \alpha d\beta, \]  

(6.20)

where

\[
\theta = \begin{bmatrix}
\frac{\partial \alpha}{A} & \frac{A_{\beta}}{AB} & -\frac{1}{R_a} \\
\frac{A_{\beta}}{AB} & \frac{\partial \alpha}{A} & 0 \\
\frac{1}{R_a} & 0 & \frac{\partial \alpha}{A} \\
\frac{\partial \beta}{B} & \frac{B_{\beta}}{AB} & -\frac{1}{R_{\beta}} \\
\frac{B_{\beta}}{AB} & \frac{\partial \beta}{B} & 0 \\
0 & \frac{1}{R_{\beta}} & \frac{\partial \beta}{B}
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
u \end{bmatrix} \\
\begin{bmatrix}
u \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
m_{11} \\
m_{21} \\
m_{31} \\
m_{12} \\
m_{22} \\
m_{32}
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
u \end{bmatrix} \\
\begin{bmatrix}
u \end{bmatrix}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial \alpha}{A} & \frac{A_{\beta}}{AB} \\
\frac{A_{\beta}}{AB} & \frac{\partial \alpha}{A} \\
\frac{1}{R_a} & 0 & \frac{\partial \alpha}{A} \\
\frac{\partial \beta}{B} & \frac{B_{\beta}}{AB} & -\frac{1}{R_{\beta}} \\
\frac{B_{\beta}}{AB} & \frac{\partial \beta}{B} & 0 \\
0 & \frac{1}{R_{\beta}} & \frac{\partial \beta}{B}
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
u \end{bmatrix} \\
\begin{bmatrix}
u \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
m_{11} \\
m_{21} \\
m_{31} \\
m_{12} \\
m_{22} \\
m_{32}
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
u \end{bmatrix} \\
\begin{bmatrix}
u \end{bmatrix}
\end{bmatrix}.
\]

(6.21)

Three types of shell panels including open cylindrical, conical and spherical shell panels are considered in this chapter. The displacements and rotation of the shell panels are shown in Figure 6.2, in which \( u, v \) and \( w \) are the displacements in axial, circumferential and radial axis, respectively, and \( u_1, v_1 \) are rotations in the directions corresponding to displacements \( v \) and \( u \). The displacement-strain relation for the cylindrical, conical and spherical shell panels are given in Eq. (6.22-6.24), respectively.
a) Cylindrical shell panels:

For cylindrical shell, we adopt the following conventions,

\[
x = x, \ y = a \cos \phi, \ z = a \sin \phi, \\
\alpha = x, \ \beta = \phi, \ A = 1, \ B = a, \ R_\alpha = \infty, \ R_\beta = a.
\] (6.22a)

Therefore, the strain components are given by

\[
\varepsilon = \begin{bmatrix}
\varepsilon_x \\
\varepsilon_\phi \\
\gamma_{\alpha\phi} \\
\gamma_{\alpha z} \\
\gamma_{\beta z} \\
\kappa_\alpha \\
\kappa_\beta \\
2\kappa_{\alpha\beta}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial_x}{a} & 0 & 0 & 0 & 0 \\
0 & \frac{\partial_\phi}{a} & -\frac{1}{a} & 0 & 0 \\
0 & 0 & \frac{\partial_x}{a} & 0 & 0 \\
0 & 0 & 0 & \frac{\partial_x}{a} & 0 \\
0 & 0 & 0 & \frac{\partial_\phi}{a} & 0 \\
0 & 0 & 0 & -\frac{\partial_x}{a} & -\frac{\partial_\phi}{a} \\
0 & 0 & 0 & 0 & -\frac{\partial_\phi}{a} \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix}
u \\
v \\
w \\
u_1 \\
v_1
\end{bmatrix} = \mathbf{B}_{xy} \cdot \mathbf{\varepsilon},
\] (6.22b)
and the gradient vector is

$$\theta = \begin{bmatrix} l_{11} \\ l_{21} \\ l_{31} \\ l_{12} \\ l_{22} \\ l_{32} \end{bmatrix} = \begin{bmatrix} \frac{\partial_x}{a} & 0 & 0 & 0 \\ 0 & \frac{\partial_y}{a} & 0 & 0 \\ 0 & 0 & \frac{1}{a} & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = C_{\psi \theta} \cdot \delta^x. \quad (6.22c)$$

b) **Conical shell panels:**

We adopt the following conventions for conical shells,

$$\alpha = s, \beta = \varphi, A = 1, B = r = s \sin \psi, B_\alpha = \sin \psi, \quad R_\alpha = \infty, R_\beta = R_\varphi = s \tan \psi. \quad (6.23a)$$

The components of the generalized strain are

$$\varepsilon = \begin{bmatrix} e_\alpha \\ e_\beta \\ \gamma_{\alpha \beta} \\ \gamma_{\alpha \psi} \\ \gamma_{\beta \psi} \\ \kappa_\alpha \\ \kappa_\beta \\ 2\kappa_{\alpha \beta} \end{bmatrix} = \begin{bmatrix} \frac{1}{s} \frac{\partial_y}{s \sin \psi} & -\frac{1}{s \tan \psi} & 0 & 0 \\ \frac{\partial_y}{s \sin \psi} & -\frac{1}{s} & 0 & 0 \\ 0 & 0 & \frac{1}{s \tan \psi} & 0 \\ 0 & 0 & -\frac{1}{s} & 0 \\ 0 & 0 & \frac{1}{s \sin \psi} & \frac{1}{s} \frac{\partial_y}{s \sin \psi} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = B_{\psi \theta} \cdot \delta^x, \quad (6.23b)$$
and the gradient vector is

\[
\begin{bmatrix}
\partial_s & 0 & 0 & 0 \\
0 & \partial_s & 0 & 0 \\
0 & 0 & \partial_s & 0 \\
0 & 0 & 0 & \partial_s \\
\end{bmatrix}
\begin{bmatrix}
u \\
v \\
w \\
v \\
\end{bmatrix}
= C_{co} \cdot \delta^e. \quad (6.23c)
\]

c) Spherical shell panels:

We adopt the following conventions for spherical shells,

\[
x = a \sin \varphi \cos \theta, \quad y = a \sin \varphi \sin \theta, \quad z = a \cos \varphi,
\]
\[
\alpha = \theta, \quad \beta = \varphi, \quad A = a \sin \varphi, \quad B = a, \quad R_\alpha = a, \quad R_\beta = a.
\]

(6.24a)
The components of the generalized strain are

\[
\varepsilon = \begin{bmatrix}
\varepsilon_\alpha \\
\varepsilon_\beta \\
\gamma_{\alpha\beta} \\
\gamma_{\alpha\theta} \\
\gamma_{\beta\theta} \\
2\kappa_{\alpha\beta}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial \phi}{a \sin \phi} & \cos \phi & -\frac{1}{a} & 0 & 0 \\
0 & \frac{\partial \phi}{a \sin \phi} & -\frac{1}{a} & 0 & 0 \\
\frac{1}{a} & 0 & \frac{\partial \phi}{a \sin \phi} & 1 & 0 \\
0 & \frac{1}{a} & \frac{\partial \phi}{a \sin \phi} & 0 & 1 \\
0 & 0 & 0 & -\frac{\partial \phi}{a \sin \phi} & -\frac{\cos \phi}{a \sin \phi} \\
0 & 0 & 0 & 0 & -\frac{\partial \phi}{a \sin \phi}
\end{bmatrix}
\begin{bmatrix}
u \\
w \\
u_1 \\
u_i \\
v_i
\end{bmatrix} = \mathbf{B}_{\varepsilon} \cdot \delta^\varepsilon,
\]

(6.24b)

and the gradient vector is

\[
\theta = \begin{bmatrix}
l_{11} \\
l_{21} \\
l_{31} \\
l_{12} \\
l_{22} \\
l_{32}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial \phi}{a \sin \phi} & \cos \phi & -\frac{1}{a} & 0 & 0 \\
\frac{-\cos \phi}{a \sin \phi} & \frac{\partial \phi}{a \sin \phi} & 0 & 0 & 0 \\
\frac{1}{a} & 0 & \frac{\partial \phi}{a \sin \phi} & 0 & 0 \\
0 & \frac{\partial \phi}{a} & 0 & 0 & 0 \\
0 & \frac{1}{a} & \frac{\partial \phi}{a} & -\frac{1}{a} & 0 \\
0 & 0 & \frac{1}{a} & \frac{\partial \phi}{a} & 0
\end{bmatrix}
\begin{bmatrix}
u \\
w \\
u_1 \\
u_i \\
v_i
\end{bmatrix} = \mathbf{C}_{\varepsilon} \cdot \delta^\varepsilon.
\]

(6.24c)


6.3 \textit{p}-elements for Shear Deformable Shell Panels

6.3.1 Shape functions

Polynomial shaped functions $f_i(\xi)$ (given in Appendix 1) have been used to solve the pre-twisted straight beams in Chapter 4, the shape functions for the problems of plates and shell panels are

\[ N_i(\xi, \eta) = f_w(\xi) f_u(\eta), \quad (6.25) \]

where

\[ m = 1, \ldots, (p_\xi + 2), ~ n = 1, \ldots, (p_\eta + 2), ~ i = 1, \ldots, (p_\xi + 2)(p_\eta + 2), \quad (6.26) \]

and $p_\xi$ and $p_\eta$ are the numbers of the polynomial terms employed in $\xi$ and $\eta$ directions, respectively. The $C^0$ polynomial functions having zero displacement at each corner node give additional degrees of freedom (DOFs) along the four edges and in the interior of the element. The DOFs of the four corner nodes are represented by $m, n \leq 2$; the DOFs along the four edges are represented if either $m$ or $n > 2$; and the DOFs in the interior are represented if both $m$ and $n > 2$.

The deflection $u, v, w$ and the rotation $u_1, v_1$ of the shell panels are interpolated by

\[
\begin{bmatrix}
    u \\
    v \\
    w \\
    u_1 \\
    v_1
\end{bmatrix} =
\begin{bmatrix}
    N_i & 0 & 0 & 0 & 0 & N_2 & 0 & 0 & 0 & 0 & N_i & 0 & 0 & 0 & 0 & \cdots \\
    0 & N_i & 0 & 0 & 0 & 0 & N_2 & 0 & 0 & 0 & 0 & N_i & 0 & 0 & 0 & \cdots \\
    0 & 0 & N_i & 0 & 0 & 0 & 0 & N_2 & 0 & 0 & 0 & 0 & N_i & 0 & 0 & \cdots \\
    0 & 0 & 0 & N_i & 0 & 0 & 0 & 0 & N_2 & 0 & 0 & 0 & 0 & N_i & 0 & \cdots \\
    0 & 0 & 0 & 0 & N_i & 0 & 0 & 0 & 0 & N_2 & 0 & 0 & 0 & 0 & N_i & \cdots 
\end{bmatrix}
\begin{bmatrix}
    d \\
\end{bmatrix}
\]

\( (6.27) \)
6.3.2 Mapping

The coordinate system used to define a cylindrical shell panel, a cut-off conical shell panel or a spherical shell panel is shown in Figure 6.3a-c, respectively.

a) To define a cylindrical shell panel, four parameters are needed: radius $a$, length $L_s$, circumferential angle $L_f$ and thickness $t$.

b) To define a cut-off conical panel, five parameters are needed: inclination angle $\psi$, initial length $s_0$, axial-length $L_s$, circumferential angle $L_f$ and thickness $t$.

c) To define a spherical panel, five parameters are needed: radius $a$, initial angle $f_0$, angle $L_f$, circumferential angle $L_s$ and thickness $t$.

Since the shell panels are mapped from rectangles with width $L_s$ and height $L_f$ (as shown in Figure 6.4) into squares with width 1, the relation between the generalized coordinates $\bar{x}, \bar{y}$ and $\bar{\xi}, \bar{\eta}$ are given in Eq. (6.28a-c):
Chapter 6: Vibration and buckling of open thin shell panels

Figure 6.4 The origin element coordinates

Cylinder:
\[
\begin{align*}
\bar{x} &= x = L_s \xi, \\
\bar{y} &= \varphi = L_f \eta, \\
\end{align*}
\]
\[
\begin{align*}
\frac{\partial x}{\partial \xi} &= \frac{\partial}{\partial \xi} / L_s, \\
\frac{\partial y}{\partial \eta} &= \frac{\partial}{\partial \eta} / L_f, \\
\end{align*}
\]
\[(6.28a)\]

Conic:
\[
\begin{align*}
\bar{x} &= s = s_0 + L_s \xi, \\
\bar{y} &= \varphi = L_f \eta, \\
\end{align*}
\]
\[
\begin{align*}
\frac{\partial x}{\partial \xi} &= \frac{\partial}{\partial \xi} / L_s, \\
\frac{\partial y}{\partial \eta} &= \frac{\partial}{\partial \eta} / L_f, \\
\end{align*}
\]
\[(6.28b)\]

Sphere:
\[
\begin{align*}
\bar{x} &= \theta = L_s \xi, \\
\bar{y} &= \varphi = f_0 + L_f \eta, \\
\end{align*}
\]
\[
\begin{align*}
\frac{\partial x}{\partial \xi} &= \frac{\partial}{\partial \xi} / L_s, \\
\frac{\partial y}{\partial \eta} &= \frac{\partial}{\partial \eta} / L_f. \\
\end{align*}
\]
\[(6.28c)\]

The Jacobian matrix is defined in terms of the Cartesian coordinates at the four corner nodes. The Jacobian matrix for the three kinds of shell panels has the same form:
\[
\mathbf{J} = \begin{bmatrix}
\frac{\partial \bar{x}}{\partial \xi} & \frac{\partial \bar{y}}{\partial \xi} \\
\frac{\partial \bar{x}}{\partial \eta} & \frac{\partial \bar{y}}{\partial \eta}
\end{bmatrix} = \begin{bmatrix} L_s & 0 \\
0 & L_f \end{bmatrix} \quad \text{and} \quad |\mathbf{J}| = L_s L_f.
\]
\[(6.29)\]
6.3.3 Stiffness, mass and geometric stiffness matrices

The equation of the dynamic stability problems of the shell panels is given by

\[
(K - \omega^2 M - \lambda G)q = 0,
\]

(6.30)

The stiffness, mass and geometric stiffness matrices of the element by applying the principle of minimum potential energy can be written as

\[
K_s^s = \iint_B B^T_s \mathbf{D}_s \cdot A_s \cdot B_s \cdot d\xi d\eta
\]

\[
= \int_0^1 \int_0^1 B^T_s (\xi, \eta) \cdot \mathbf{D}_s \cdot B_s (\xi, \eta) \cdot |J| \cdot A_s (\xi, \eta) \cdot B_s (\xi, \eta) \cdot d\xi d\eta,
\]

(6.31)

\[
M_s^s = \rho \iint_B N^T_s \mathbf{N} \cdot A_s \cdot B_s \cdot d\xi d\eta
\]

\[
= \rho \int_0^1 \int_0^1 N^T_s (\xi, \eta) \cdot \mathbf{N} (\xi, \eta) \cdot |J| \cdot A_s (\xi, \eta) \cdot B_s (\xi, \eta) \cdot d\xi d\eta,
\]

(6.32)

\[
G_s^s = \iint_B C^T_s X_s \cdot A_s \cdot B_s \cdot d\xi d\eta
\]

\[
= \int_0^1 \int_0^1 C^T_s (\xi, \eta) \cdot X \cdot C_s (\xi, \eta) \cdot |J| \cdot A_s (\xi, \eta) \cdot B_s (\xi, \eta) \cdot d\xi d\eta,
\]

(6.33)

where the subscript ‘s’ on the matrices \(B_s, C_s\) and value \(A_s, B_s\) denotes the corresponding matrices \(B_{cy}, B_{co}, B_{sp}\) or \(C_{cy}, C_{co}, C_{sp}\), and \(A_{cy}, A_{co}, A_{sp}\) or \(B_{cy}, B_{co}, B_{sp}\) of cylindrical, conical and spherical shells panels, respectively.

The coefficients of the stiffness, mass and geometric matrices due to axial
compression of cylindrical shell panels are given in Appendix 6. The entries of the stiffness and mass matrices of conical and spherical shell panels are given in Appendix 7 and 8, respectively.

### 6.3.4 Integration procedures

When using polynomial series, the integration forms appeared in the calculation of the element matrices of cylindrical panels are:

\[
\int_0^1 \xi^i \xi^j d\xi, \quad \int_0^1 \eta^i \eta^j d\xi, \quad i=0,1,2,\cdots p+1, \quad j=0,1,2,\cdots p+1.
\]

The integration forms appeared in the calculation of the element matrices of the cut-off conical shell panels are:

\[
\int_0^1 \xi^i \xi^j d\xi, \quad \int_0^1 \xi^i \xi^j d\xi, \quad \int_0^1 \xi^i \xi^j d\xi, \quad \int_0^1 \eta^i \eta^j d\xi, \quad \int_0^1 \eta^i \eta^j d\xi,
\]

where \( s = s_0 + L_\xi \), \( i=0,1,2,\cdots p+1, \quad j=0,1,2,\cdots p+1 \).

The integration forms appeared in the calculation of the element matrices of spherical shell panels are:

\[
\int_0^1 (\sin \varphi) \eta^i \eta^j d\eta, \quad \int_0^1 (\cos \varphi) \eta^i \eta^j d\eta, \quad \int_0^1 \cos \varphi \eta^i \eta^j d\eta, \quad \int_0^1 \cos \varphi \eta^i \eta^j d\eta,
\]

where \( \varphi = f_0 + L, \eta, \quad i=0,1,2,\cdots p+1, \quad j=0,1,2,\cdots p+1 \).

The terms listed above can be integrated analytically and the symbolic toolbox in MATLAB is adopted for the integration procedure.
6.4 Natural Frequencies of Shell Panels

6.4.1 Convergence study and compared with existed solutions

Table 6.1 presents a convergence study using different polynomial terms \( p \) for a fully clamped square planform, cylindrical thin shallow shell panel with the following geometry: \( t/L_s = 0.01 \) or 0.05, \( a/L_s = 2 \), \( L_f = 2\arcsin[L_s/(2a)] \) and Poisson’s ratio \( \nu \) is 0.3 which have been considered by Lim [184]. The frequency parameters \( \Omega = \omega(L_s)^2(\rho t/D)^{1/2} \) are compared with the solutions of Lim [184] and ANSYS using SHELL63 element with 100\times50 meshes. It is found in Table 6.1 that when \( t/L_s = 0.01 \), the present results fit better with those of ANSYS when compared with those of Lim’s [184]. However, when the thickness ratio increased to \( t/L_s = 0.05 \), the frequency parameters are lower than the results of Lim [184] and ANSYS. This may be caused by the assumption that the distance \( \varsigma \) away from the mid-surface is assumed to be much smaller than the curvature radius \( R \) (as shown in Eq. (6.1)). So in this chapter only thin shell panels are studied. The thickness ratio of the shell panels is chosen as 0.01 if not specified otherwise.

Table 6.2 presents a convergence and comparison study of frequency parameters

\[
\Omega = \omega L_s^2 \sqrt{\rho t/D}
\]

for a fully clamped thin conical shell panel with \( \nu = 0.3 \), \( s_0/L_s = 2/3 \), \( t/L_s = 0.01 \), \( \psi = \pi/6 \) and \( L_f = \pi/3 \) which have been considered by Lim [191] and Srinivasan and Krishnan [188]. It is shown in Table 6.2 that when \( p = 12 \) the frequency parameters fit better with the results of NASTRAN [191] when compared with results of Lim [191], Srinivasan and Krishnan [188] and Cheung et al [189]. Therefore, the accuracy of the present method on conical shell panels is established.

Table 6.3 gives a convergence study for frequency parameters

\[
\Omega = \omega b \sqrt{\rho / E}
\]
a fully clamped spherical shell panels with square planform of width \( b \) which has been considered by Liew [197]. Since the present spherical shell panels are obtained by rotating an arc along the central axis with an angle \( L_s \), the spherical shell panels cannot be shaped exactly with square planform and has a little difference from Liew’s [197]. To be close to a spherical shell panel with the square planform of width \( b \), the geometrics of the shell panels are chosen as follows: \( t/b=0.01, a/b=2, f_0=\arccos(0.5b/a), L_f=2\arcsin(0.5b/a), L_s=2\arcsin(0.5b/a) \).

The frequency parameters are compared well with results of Liew [197] and those of ANSYS as shown in Table 6.3. It is found from Table 6.3 that when \( p=12 \) the present results agree well with results of ANSYS and Liew [197].

Table 6.1 Frequency parameters \( \Omega=\omega(L_s)^2(\rho t/D)^{1/2} \) convergence comparison for CCCC cylindrical shallow shell panels with thickness ratio \( t/L_s=0.01 \) and \( 0.05 \).

<table>
<thead>
<tr>
<th>( t/L_s )</th>
<th>( p )</th>
<th>Mode number</th>
</tr>
</thead>
<tbody>
<tr>
<td>&amp; 1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>0.01</td>
<td>4</td>
<td>106.9425</td>
</tr>
<tr>
<td>&amp; 6</td>
<td>98.6916</td>
<td>119.1810</td>
</tr>
<tr>
<td>&amp; 8</td>
<td>98.4554</td>
<td>117.2680</td>
</tr>
<tr>
<td>&amp; 10</td>
<td>98.4524</td>
<td>117.2113</td>
</tr>
<tr>
<td>&amp; 12</td>
<td>98.4522</td>
<td>117.2105</td>
</tr>
<tr>
<td>ANSYS</td>
<td>98.5450</td>
<td>117.4345</td>
</tr>
<tr>
<td>Lim [184]</td>
<td>99.263</td>
<td>119.00</td>
</tr>
<tr>
<td>0.05</td>
<td>4</td>
<td>45.2160</td>
</tr>
<tr>
<td>&amp; 6</td>
<td>45.1155</td>
<td>69.6151</td>
</tr>
<tr>
<td>&amp; 8</td>
<td>45.1101</td>
<td>69.5887</td>
</tr>
<tr>
<td>&amp; 10</td>
<td>45.1089</td>
<td>69.5850</td>
</tr>
<tr>
<td>&amp; 12</td>
<td>45.1085</td>
<td>69.5841</td>
</tr>
<tr>
<td>ANSYS</td>
<td>45.8480</td>
<td>72.8682</td>
</tr>
<tr>
<td>Lim [184]</td>
<td>46.241</td>
<td>74.300</td>
</tr>
</tbody>
</table>
Table 6.2 Frequency convergence and comparison of $\Omega = \omega L_s^2 \sqrt{\rho t / D}$ for a fully clamped thin conical shell panel with $v=0.3$, $s_0/L_s=2/3$, $t/L_s=0.01$, $\psi=\pi/6$ and $L_f=\pi/3$.

<table>
<thead>
<tr>
<th>$p$</th>
<th>Mode number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>237.9069</td>
<td>368.0849</td>
<td>388.6512</td>
<td>498.464</td>
<td>1229.0155</td>
<td>1261.6709</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>210.0154</td>
<td>261.3068</td>
<td>308.5157</td>
<td>355.8016</td>
<td>404.0171</td>
<td>416.5947</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>209.1421</td>
<td>255.3766</td>
<td>306.5548</td>
<td>349.3306</td>
<td>396.4531</td>
<td>402.0397</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>209.1260</td>
<td>255.1827</td>
<td>306.4862</td>
<td>349.1354</td>
<td>396.0305</td>
<td>400.4320</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>209.1252</td>
<td>255.1792</td>
<td>306.4817</td>
<td>349.1281</td>
<td>395.9821</td>
<td>400.3609</td>
<td></td>
</tr>
<tr>
<td></td>
<td>ANSYS</td>
<td>209.5959</td>
<td>256.2837</td>
<td>307.2732</td>
<td>350.6870</td>
<td>396.9930</td>
<td>400.6634</td>
</tr>
<tr>
<td></td>
<td>Nastran [191]</td>
<td>208.99</td>
<td>254.84</td>
<td>305.83</td>
<td>348.22</td>
<td>394.54</td>
<td>399.87</td>
</tr>
<tr>
<td></td>
<td>Lim [191]</td>
<td>209.73</td>
<td>256.55</td>
<td>307.80</td>
<td>351.26</td>
<td>400.00</td>
<td>401.37</td>
</tr>
<tr>
<td></td>
<td>Srinivasan [188]</td>
<td>202.7</td>
<td>260.1</td>
<td>305.6</td>
<td>355.0</td>
<td>402.9</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>Cheung [189]</td>
<td>213.4</td>
<td>262.5</td>
<td>314.7</td>
<td>358.6</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 6.3 Convergence and comparison of $\Omega = \omega b \sqrt{\rho / E}$ for a fully clamped shallow spherical shell panels with square planform of width $b$, and $t/b=0.01$, $a/b=2$, $f_0=\arccos(0.5b/a)$, $L_f=2\arcsin(0.5b/a)$, $L_s=2\arcsin(0.5b/a)$.

<table>
<thead>
<tr>
<th>$t/b$</th>
<th>$p$</th>
<th>Mode number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>4</td>
<td>0.59498</td>
<td>0.59548</td>
<td>0.65257</td>
<td>0.70296</td>
<td>3.54864</td>
<td>3.56181</td>
<td></td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0.57678</td>
<td>0.57678</td>
<td>0.59327</td>
<td>0.63058</td>
<td>0.66040</td>
<td>0.73671</td>
<td></td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>0.57629</td>
<td>0.57632</td>
<td>0.59117</td>
<td>0.63017</td>
<td>0.64776</td>
<td>0.72470</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.57629</td>
<td>0.57631</td>
<td>0.59111</td>
<td>0.63017</td>
<td>0.64729</td>
<td>0.72418</td>
<td></td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>0.57629</td>
<td>0.57631</td>
<td>0.59111</td>
<td>0.63016</td>
<td>0.64728</td>
<td>0.72417</td>
<td></td>
</tr>
<tr>
<td>ANSYS</td>
<td></td>
<td>0.57610</td>
<td>0.57610</td>
<td>0.59090</td>
<td>0.63040</td>
<td>0.64710</td>
<td>0.72490</td>
<td></td>
</tr>
<tr>
<td>Liew [197]</td>
<td></td>
<td>0.57638</td>
<td>0.57638</td>
<td>0.59134</td>
<td>0.63038</td>
<td>0.64764</td>
<td>0.72609</td>
<td></td>
</tr>
<tr>
<td>Liew [205]</td>
<td></td>
<td>0.58099</td>
<td>0.58099</td>
<td>0.59594</td>
<td>0.63537</td>
<td>0.65422</td>
<td>0.73299</td>
<td></td>
</tr>
</tbody>
</table>

6.4.2 Natural frequencies of cylindrical shell panels

With the accuracy of the present method established application of the polynomial $p$-element to the thin, open, shallow cylindrical shell panels with fully clamped
(CCCC) boundary condition are considered in this section. Numerical results are given for the non-dimensional parameter $\Omega = \alpha a \sqrt{\rho/E}$ with radius $a=1\text{m}$, thickness ratio $t/a=0.01$ as shown in Table 6.4. The frequency parameters of the cylindrical shell panels with the aspect ratio $L_s/a$ varied from 0.5 to 5.0 and the circumferential angle $L_f$ ranging from $\pi/6$ to $\pi/2$ are considered. As shown in Table 6.4, the frequency parameters decreased by the axial length $L_s$ and the circumferential angle $L_f$ separately. The frequencies become lower when the size of the cylindrical panels is increased in the circumferential and axial directions. Moreover, when $L_f=\pi/6$, $L_s/a$ increased from 0.5 to 5, the first mode of the frequency parameter is decreased from 0.9186 to 0.6707, while when $L_f=\pi/2$, the frequency parameter decreased from 0.6258 to 0.1324 with the axial length ratio varying from 0.5 to 5. It shows that when the cylindrical shell panel has a larger circumferential angle, the frequency will decrease more rapidly by the axial length $L_s$. Similarly, when the cylindrical shell panel has a larger axial length $L_s$, the frequency will decrease faster by the circumferential angle $L_f$, too.

6.4.3 Natural frequencies of open conical shell panels

Numerical results are obtained for thin, open, shallow, conical shell panels with fully clamped (CCCC) boundary conditions in this section. The frequency parameter is defined as $\Omega = \alpha \sqrt{\rho/E}$. Five independent parameters including inclination angle $\psi$, initial length $s_0$, axial length $L_s$, circumferential angle $L_f$ and thickness $h$ are needed to define the geometry of a conical shell panel. Therefore, firstly, similar to the cylindrical shell panels, the thickness $t$ is chosen as 0.01m. Then, the inclination angle $\psi=\pi/6$ and $\pi/3$ are chosen for every value of $s_0=0.5\text{m}$ and 1m. Finally, by keeping $s_0=0.5\text{m}$ or 1m, the size of the panels are enlarged by choosing $L_s$ ranging from 0.5m to 5m and the circumferential angle $L_f$ varying from $\pi/6$ to $\pi/2$. The frequency parameters corresponding to $\psi=\pi/6$ with $s_0=0.5\text{m}$
and 1m are given in Table 6.5a and 6.5b, respectively. And the frequency parameters when \( \psi = \pi/3 \), \( s_0 = 0.5 \text{m} \) and 1m are obtained in Table 6.6a and 6.6b, respectively. As shown in Table 6.5 and 6.6, similar to the cylindrical shell panels, the frequencies are decreased by the length \( L_s \) and \( L_f \) separately. Moreover, taking \( \psi = \pi/6, s_0 = 0.5 \text{m}, L_f = \pi/2 \) as an example as shown in Table 6.5a, when \( L_s \) increased from 0.5m to 5m, the first mode of the frequency parameters varied from 1.2113 to 0.0812, which decreased almost to only 6.7% of the origin frequency, which shows that the axial length \( L_s \) has a very notable influence on the frequency of the conical shell panels when compared with other shape control parameters of the conical shell panels. Similar phenomenon can be found in Table 6.5b and 6.6a-6.6b. As shown in Table 6.5a and 6.5b, when the \( \psi = \pi/6 \) and the conical shell panel having the same length \( L_s \) and \( L_f \), the frequency with \( s_0 = 0.5 \text{m} \) are larger than those of \( s_0 = 1 \text{m} \), which means the frequency parameter of the cut-off conical shell panels can be increased by choosing the cutting-off length \( s_0 \) close to the conical tip. The influence of the inclination angle \( \psi \) can be found from the difference between Table 6.5a and 6.6a or from Table 6.5b and 6.6b, which shows that the conical shell panels having a smaller inclination angle will get a higher frequency.

### 6.4.4 Natural frequencies of open spherical shell panels

With the convergence study given in Section 6.4.1 for the spherical shell panels, the frequency parameters defined as \( \Omega = \omega a \sqrt{\rho/E} \) with CCCC boundary conditions are studied. The vibration problems of spherical shell panels with radius \( a = 1 \text{m} \) and thickness ratio \( t/a = 0.01 \) are chosen for study. The initial angle \( f_0 \) is taking as \( \pi/6 \) and \( \pi/4 \), and \( L_f \) are chosen to vary from \( \pi/6 \) to \( \pi/2 \). The circumferential angle \( L_s \) is ranging from \( \pi/6 \) to \( \pi/2 \). The results with \( f_0 = \pi/6 \) and \( \pi/4 \) are given in Table 6.7 and Table 6.8, respectively. Similar to cylindrical and conical shell panels, the frequency parameters are decreased with circumferential
angle $L_s$ and axial angle $L_f$ separately. As shown in Table 6.7 and 6.8, when $L_f$ and $L_s$ are kept constant, the frequency with initial angle $f_0=\pi/6$ are higher than those with $f_0=\pi/4$. Moreover, when the circumferential angle ($L_s$ for spherical shell panels or $L_f$ for cylindrical and conical shell panels) ranged from $\pi/6$ to $\pi/2$, for a spherical shell panel with $f_0=\pi/6$, $L_f=\pi/4$, the frequency decreased from 1.3054 to 1.0442, while for a cylindrical panel with $L_s=1$m, the frequency reduced from 0.7308 to 0.3458, for a conical shell panel with $\psi=\pi/6$, $s_0=0.5$m, $L_s=1$m, the frequency decreased from 1.3832 to 0.6086, which shows that the influence of the circumferential angle on the spherical shell panels are much weaker than those of cylindrical and conical shell panels. Similar phenomenon can be found by comparing the frequency parameters of the shell panels with other values of circumferential angle $L_f$ or $L_s$. Moreover, contrary to the axial length $L_s$ of the cylindrical and conical shell panels which can be increased infinitely, the axial length $L_f$ of the spherical shell panels must less than $\pi$, so that the frequency of spherical shell panels will not be impacted seriously by the axial length $L_f$. And the frequencies of spherical shell panels are generally higher than the frequencies of cylindrical and conical shell panels when the shell panels with similar geometric dimensions.

### 6.4.5 Vibration mode shapes of open cylindrical, conical and spherical shell panels

The frequency parameters $\Omega = \omega \sqrt{\rho/E}$ and the mode shapes of the three displacements $u$, $v$ and $w$ of cylindrical, conical and spherical shell panels comparing with results of ANSYS are given in Table 6.9a-c, respectively. The geometry of the cylindrical shell panel is chosen as follows: $a=1$m, $t/a=0.01$, $L_s/a=1$, $L_f=\pi/2$. The geometry of the conical shell panel is: $\psi=\pi/6$, $t=0.01$m, $s_0=1$m, $L_s=1$m, $L_f=\pi/2$. The dimension of the spherical shell panel is: $a=1$m, $t/a=0.01$, $L_s=1$m, $L_f=\pi/2$. The initial angle $f_0=\pi/6$, $L_f=\pi/4$, the frequency decreased from 1.3054 to 1.0442, while for a cylindrical panel with $L_s=1$m, the frequency reduced from 0.7308 to 0.3458, for a conical shell panel with $\psi=\pi/6$, $s_0=0.5$m, $L_s=1$m, the frequency decreased from 1.3832 to 0.6086, which shows that the influence of the circumferential angle on the spherical shell panels are much weaker than those of cylindrical and conical shell panels. Similar phenomenon can be found by comparing the frequency parameters of the shell panels with other values of circumferential angle $L_f$ or $L_s$. Moreover, contrary to the axial length $L_s$ of the cylindrical and conical shell panels which can be increased infinitely, the axial length $L_f$ of the spherical shell panels must less than $\pi$, so that the frequency of spherical shell panels will not be impacted seriously by the axial length $L_f$. And the frequencies of spherical shell panels are generally higher than the frequencies of cylindrical and conical shell panels when the shell panels with similar geometric dimensions.
As shown in Table 6.10a-c, the frequency parameters and vibration mode shapes of the three kinds of shell panels are agreed well with those of ANSYS, except the mode shapes in radial direction (i.e. $u$ displacement) of the conical shell panel. The reason for this is that the displacements obtained by ANSYS are shown in a spherical coordinate system, which is not agreed with the present conical coordinate system in $u$ direction. As shown in Table 6.9a-c, the contours form close loops in most cases, but they can also enclose several small loops, one example is shown in Figure 6.9b of the displacement $U_w$ in mode 4.

### 6.5 Buckling Problems of Open Cylindrical Shell Panels

#### 6.5.1 Convergence study and compared with results of ANSYS

Table 6.10 presents a convergence study using different polynomial terms $p$ for a cylindrical shell panel clamped on one circumferential side and free otherwise (CFFF) subject to axially compressed loads with the following geometry: $a=1$m, $t/a=0.01$, $L_s/a=1$, $L_f=\pi/2$ and Poisson’s ratio $\nu=0.3$. The buckling load intensity factors $\lambda = N_c a \sqrt{3(1-\nu^2)}/(Et^2)$ are compared with the solutions of ANSYS using SHELL63 element with $100\times50$ meshes. It is found in Table 6.10 that when $p=12$ the present frequencies and buckling modes compared well with results of ANSYS.
Chapter 6: Vibration and buckling of open thin shell panels

Table 6.10a Convergence and comparison of buckling load intensity factors

\[ \lambda = \frac{N_c a \sqrt{3(1 - \nu^2) / (E t^2)}}{2231} \]

for a CFFF cylindrical shell panel

<table>
<thead>
<tr>
<th>t/a</th>
<th>p</th>
<th>Mode number</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>0.01</td>
<td>4</td>
<td>0.0997</td>
</tr>
<tr>
<td>6</td>
<td>0.0952</td>
<td>0.1090</td>
</tr>
<tr>
<td>8</td>
<td>0.0936</td>
<td>0.1066</td>
</tr>
<tr>
<td>10</td>
<td>0.0934</td>
<td>0.1063</td>
</tr>
<tr>
<td>12</td>
<td>0.0933</td>
<td>0.1062</td>
</tr>
<tr>
<td>ANSYS</td>
<td>0.0937</td>
<td>0.1067</td>
</tr>
</tbody>
</table>

Table 6.10b Buckling mode shape comparison of a CFFF cylindrical shell panel with results of ANSYS

<table>
<thead>
<tr>
<th>Deflection amplitude</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>U_u</td>
</tr>
<tr>
<td></td>
<td>U_v</td>
</tr>
<tr>
<td></td>
<td>U_w</td>
</tr>
<tr>
<td>Present p=12</td>
<td></td>
</tr>
<tr>
<td>ANSYS</td>
<td></td>
</tr>
</tbody>
</table>

6.5.2 Buckling loads of open cylindrical shell panels

With the convergence study presented in Section 6.5.1, the buckling loads intensity factors of cylindrical shell panels are studied in this section. Fully clamped (CCCC) cylindrical shell panels and clamped on one of the circumferential side (CFFF) are considered. The thickness ratio is chosen as t/a=0.01, Ls/a is varied from 0.5 to 5 and circumferential angle Lf is changed from \( \pi/6 \) to \( \pi/2 \). The buckling load intensity factors \( \lambda = N_c a \sqrt{3(1 - \nu^2) / (E t^2)} \) of the cylindrical shell panels are given in Table 6.11. As shown in Table 6.11, for CFFF boundary...
condition the buckling load intensity factors are increased with the circumferential angle $L_f$ only when the aspect ratio $L_s/a=1$. Meanwhile, for CFFF cylindrical shell panels, the buckling loads factors decreased by the aspect ratio monotonically. The buckling load factors of fully clamped cylindrical shell panels decreased by the circumferential angle, but not varied monotonically with the aspect ratio.

6.6 Natural Frequencies of Shell Panels with Larger Circumferential Angle and/or Thinner Thickness

The numerical examples given in Section 6.4 and 6.5 are for open thin shallow shell panels with thickness $t=0.01m$. In this section, the polynomial $p$-elements are extended to solve open shell panels with larger circumferential angle and/or thinner thicknesses. The natural frequencies of cylindrical, conical and spherical shell panels are obtained by increasing the polynomial terms to achieve excellent agreement with results of ANSYS. The frequencies of cylindrical, conical and spherical shell panels with circumferential angle $\pi/2$ or $\pi$, thickness $t=0.001m$ or $0.01m$ are taken for comparison as shown in Table 6.12-6.14, respectively. Besides, the relative errors of the frequencies compared with those obtained by ANSYS are underlined in the tables. Only one element is used in all the cases. As shown in Table 6.12-6.14, when the shell panels have larger circumferential angle ($\pi$) or thinner thickness (0.001), much more polynomial terms ($p$ reach to 28) are needed to achieve good agreement with results of ANSYS. When the shell panels become thinner or deeper, the number of the natural modes in the circumferential direction will increase a lot, which means the modal density of the mapped square region with width 1 are increased for a specified frequency range. To get more accurate information on the density populated modal density region, higher order
polynomial terms are needed. As stated by Zhu [206], the effective number of digits of the resulting integration will decreased rapidly by the high order polynomial terms, and the polynomial $p$-elements will become numerical unstable. To overcome this problem, the symbolic tool of MATLAB is employed in the integration procedure, and the immediate integration results are kept as symbolic value until the integration procedure is finally completed. However, longer computational time and more memory are required for the symbolic operation, especially for the spherical shell panels, and then followed by conical and cylindrical shell panels.
Table 6.4 Frequency parameters $\Omega = \sqrt{\frac{1}{\rho E} a \omega}$ for fully clamped, thin, open cylindrical shell panels with $a=1$m and $\nu=0.01$

<table>
<thead>
<tr>
<th>$L_s/a$</th>
<th>$L_f$</th>
<th>Mode number</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>0.5</td>
<td>$\pi/6$</td>
<td>0.9186</td>
</tr>
<tr>
<td></td>
<td>$\pi/4$</td>
<td>0.7286</td>
</tr>
<tr>
<td></td>
<td>$\pi/3$</td>
<td>0.6653</td>
</tr>
<tr>
<td></td>
<td>$\pi/2$</td>
<td>0.6285</td>
</tr>
<tr>
<td>1</td>
<td>$\pi/6$</td>
<td>0.7308</td>
</tr>
<tr>
<td></td>
<td>$\pi/4$</td>
<td>0.4595</td>
</tr>
<tr>
<td></td>
<td>$\pi/3$</td>
<td>0.4044</td>
</tr>
<tr>
<td></td>
<td>$\pi/2$</td>
<td>0.3458</td>
</tr>
<tr>
<td>2</td>
<td>$\pi/6$</td>
<td>0.6828</td>
</tr>
<tr>
<td></td>
<td>$\pi/4$</td>
<td>0.3402</td>
</tr>
<tr>
<td></td>
<td>$\pi/3$</td>
<td>0.2688</td>
</tr>
<tr>
<td></td>
<td>$\pi/2$</td>
<td>0.2088</td>
</tr>
<tr>
<td>5</td>
<td>$\pi/6$</td>
<td>0.6707</td>
</tr>
<tr>
<td></td>
<td>$\pi/4$</td>
<td>0.3010</td>
</tr>
<tr>
<td></td>
<td>$\pi/3$</td>
<td>0.1818</td>
</tr>
<tr>
<td></td>
<td>$\pi/2$</td>
<td>0.1324</td>
</tr>
</tbody>
</table>
Table 6.5a Frequency parameters $\Omega = \alpha \sqrt{\rho / E}$ for fully clamped, thin, open conical shell panels with $\psi = \pi / 6$, $s_0 = 0.5$ m and $t = 0.01$ m

<table>
<thead>
<tr>
<th>$L_s$</th>
<th>$L_f$</th>
<th>Mode number</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>0.5</td>
<td>$\pi / 6$</td>
<td>2.4194</td>
</tr>
<tr>
<td></td>
<td>$\pi / 4$</td>
<td>2.0002</td>
</tr>
<tr>
<td></td>
<td>$\pi / 3$</td>
<td>1.4751</td>
</tr>
<tr>
<td></td>
<td>$\pi / 2$</td>
<td>1.2113</td>
</tr>
<tr>
<td>1</td>
<td>$\pi / 6$</td>
<td>1.3832</td>
</tr>
<tr>
<td></td>
<td>$\pi / 4$</td>
<td>1.0006</td>
</tr>
<tr>
<td></td>
<td>$\pi / 3$</td>
<td>0.7461</td>
</tr>
<tr>
<td></td>
<td>$\pi / 2$</td>
<td>0.6086</td>
</tr>
<tr>
<td>2</td>
<td>$\pi / 6$</td>
<td>0.6829</td>
</tr>
<tr>
<td></td>
<td>$\pi / 4$</td>
<td>0.4154</td>
</tr>
<tr>
<td></td>
<td>$\pi / 3$</td>
<td>0.3374</td>
</tr>
<tr>
<td></td>
<td>$\pi / 2$</td>
<td>0.2692</td>
</tr>
<tr>
<td>5</td>
<td>$\pi / 6$</td>
<td>0.1659</td>
</tr>
<tr>
<td></td>
<td>$\pi / 4$</td>
<td>0.1191</td>
</tr>
<tr>
<td></td>
<td>$\pi / 3$</td>
<td>0.1002</td>
</tr>
<tr>
<td></td>
<td>$\pi / 2$</td>
<td>0.0812</td>
</tr>
</tbody>
</table>
Table 6.5b Frequency parameters \( \Omega = \alpha \sqrt{\rho / E} \) for fully clamped, thin, open conical shell panels with \( \psi = \pi / 6 \), \( s_0 = 1 \text{m} \) and \( t = 0.01 \text{m} \)

<table>
<thead>
<tr>
<th>( L_s )</th>
<th>( L_f )</th>
<th>Mode number</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 3 )</th>
<th>( 4 )</th>
<th>( 5 )</th>
<th>( 6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>( \pi / 6 )</td>
<td>1.3844</td>
<td>1.6941</td>
<td>1.7990</td>
<td>2.2158</td>
<td>2.3393</td>
<td>2.9929</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \pi / 4 )</td>
<td>1.0538</td>
<td>1.1960</td>
<td>1.5243</td>
<td>1.5966</td>
<td>1.7224</td>
<td>2.0495</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \pi / 3 )</td>
<td>0.8986</td>
<td>0.9712</td>
<td>1.3830</td>
<td>1.4395</td>
<td>1.4651</td>
<td>1.5452</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \pi / 2 )</td>
<td>0.7873</td>
<td>0.8108</td>
<td>1.0552</td>
<td>1.0891</td>
<td>1.3678</td>
<td>1.3761</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>( \pi / 6 )</td>
<td>0.9576</td>
<td>1.0387</td>
<td>1.1137</td>
<td>1.2756</td>
<td>1.3338</td>
<td>1.4461</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \pi / 4 )</td>
<td>0.6068</td>
<td>0.7800</td>
<td>0.8518</td>
<td>0.9726</td>
<td>1.0863</td>
<td>1.0913</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \pi / 3 )</td>
<td>0.4963</td>
<td>0.5609</td>
<td>0.7450</td>
<td>0.7852</td>
<td>0.9046</td>
<td>0.9650</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \pi / 2 )</td>
<td>0.4078</td>
<td>0.436</td>
<td>0.6083</td>
<td>0.6147</td>
<td>0.6808</td>
<td>0.6819</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>( \pi / 6 )</td>
<td>0.4873</td>
<td>0.5746</td>
<td>0.6337</td>
<td>0.6624</td>
<td>0.7471</td>
<td>0.7672</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \pi / 4 )</td>
<td>0.3079</td>
<td>0.3885</td>
<td>0.4382</td>
<td>0.5129</td>
<td>0.5502</td>
<td>0.6266</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \pi / 3 )</td>
<td>0.2585</td>
<td>0.2762</td>
<td>0.3662</td>
<td>0.3931</td>
<td>0.4556</td>
<td>0.4818</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \pi / 2 )</td>
<td>0.2048</td>
<td>0.2173</td>
<td>0.2883</td>
<td>0.2989</td>
<td>0.3456</td>
<td>0.3532</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>( \pi / 6 )</td>
<td>0.1429</td>
<td>0.1919</td>
<td>0.1946</td>
<td>0.2387</td>
<td>0.2391</td>
<td>0.2807</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \pi / 4 )</td>
<td>0.1044</td>
<td>0.1134</td>
<td>0.1469</td>
<td>0.1568</td>
<td>0.1826</td>
<td>0.1969</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \pi / 3 )</td>
<td>0.0870</td>
<td>0.0921</td>
<td>0.1145</td>
<td>0.1266</td>
<td>0.1493</td>
<td>0.1506</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \pi / 2 )</td>
<td>0.0712</td>
<td>0.0714</td>
<td>0.0894</td>
<td>0.0977</td>
<td>0.1163</td>
<td>0.1204</td>
<td></td>
</tr>
</tbody>
</table>
Table 6.6a Frequency parameters $\Omega = \sqrt{\rho / E}$ for fully clamped, thin, open conical shell panels with $\psi = \pi / 3$, $s_0 = 0.5$ m and $t = 0.01$ m

<table>
<thead>
<tr>
<th>$L_\psi$</th>
<th>$L_f$</th>
<th>Mode number</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>0.5</td>
<td>$\pi / 6$</td>
<td>0.9524</td>
</tr>
<tr>
<td></td>
<td>$\pi / 4$</td>
<td>0.7828</td>
</tr>
<tr>
<td></td>
<td>$\pi / 3$</td>
<td>0.6926</td>
</tr>
<tr>
<td></td>
<td>$\pi / 2$</td>
<td>0.6095</td>
</tr>
<tr>
<td>1</td>
<td>$\pi / 6$</td>
<td>0.5267</td>
</tr>
<tr>
<td></td>
<td>$\pi / 4$</td>
<td>0.4266</td>
</tr>
<tr>
<td></td>
<td>$\pi / 3$</td>
<td>0.3378</td>
</tr>
<tr>
<td></td>
<td>$\pi / 2$</td>
<td>0.2865</td>
</tr>
<tr>
<td>2</td>
<td>$\pi / 6$</td>
<td>0.2698</td>
</tr>
<tr>
<td></td>
<td>$\pi / 4$</td>
<td>0.1801</td>
</tr>
<tr>
<td></td>
<td>$\pi / 3$</td>
<td>0.1561</td>
</tr>
<tr>
<td></td>
<td>$\pi / 2$</td>
<td>0.1296</td>
</tr>
<tr>
<td>5</td>
<td>$\pi / 6$</td>
<td>0.0689</td>
</tr>
<tr>
<td></td>
<td>$\pi / 4$</td>
<td>0.0545</td>
</tr>
<tr>
<td></td>
<td>$\pi / 3$</td>
<td>0.0459</td>
</tr>
<tr>
<td></td>
<td>$\pi / 2$</td>
<td>0.0398</td>
</tr>
</tbody>
</table>
Table 6.6b Frequency parameters $\Omega=\alpha\sqrt{\rho/E}$ for fully clamped, thin, open conical shell panels with $\psi=\pi/3$, $s_0=1$ m and $t=0.01$ m

<table>
<thead>
<tr>
<th>$L_s$</th>
<th>$L_f$</th>
<th>Mode number</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>0.5</td>
<td>$\pi/6$</td>
<td>0.5612</td>
</tr>
<tr>
<td></td>
<td>$\pi/4$</td>
<td>0.5108</td>
</tr>
<tr>
<td></td>
<td>$\pi/3$</td>
<td>0.4733</td>
</tr>
<tr>
<td></td>
<td>$\pi/2$</td>
<td>0.4485</td>
</tr>
<tr>
<td>1</td>
<td>$\pi/6$</td>
<td>0.3589</td>
</tr>
<tr>
<td></td>
<td>$\pi/4$</td>
<td>0.2675</td>
</tr>
<tr>
<td></td>
<td>$\pi/3$</td>
<td>0.2402</td>
</tr>
<tr>
<td></td>
<td>$\pi/2$</td>
<td>0.2114</td>
</tr>
<tr>
<td>2</td>
<td>$\pi/6$</td>
<td>0.1971</td>
</tr>
<tr>
<td></td>
<td>$\pi/4$</td>
<td>0.1353</td>
</tr>
<tr>
<td></td>
<td>$\pi/3$</td>
<td>0.1208</td>
</tr>
<tr>
<td></td>
<td>$\pi/2$</td>
<td>0.1023</td>
</tr>
<tr>
<td>5</td>
<td>$\pi/6$</td>
<td>0.0597</td>
</tr>
<tr>
<td></td>
<td>$\pi/4$</td>
<td>0.0480</td>
</tr>
<tr>
<td></td>
<td>$\pi/3$</td>
<td>0.0402</td>
</tr>
<tr>
<td></td>
<td>$\pi/2$</td>
<td>0.0350</td>
</tr>
</tbody>
</table>
Table 6.7 Frequency parameters $\Omega = \omega_0 \sqrt{\rho / E}$ for fully clamped, thin, open spherical shell panels with $a=1m$, $f_0=\pi/6$, and $t/a=0.01$

<table>
<thead>
<tr>
<th>$L_f$</th>
<th>$L_s$</th>
<th>Mode number</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>$\pi/6$</td>
<td>$\pi/6$</td>
<td>1.4826</td>
</tr>
<tr>
<td>$\pi/4$</td>
<td></td>
<td>1.3020</td>
</tr>
<tr>
<td>$\pi/3$</td>
<td></td>
<td>1.2168</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td></td>
<td>1.1625</td>
</tr>
<tr>
<td>$\pi/4$</td>
<td>$\pi/6$</td>
<td>1.3054</td>
</tr>
<tr>
<td>$\pi/4$</td>
<td></td>
<td>1.1593</td>
</tr>
<tr>
<td>$\pi/3$</td>
<td></td>
<td>1.0971</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td></td>
<td>1.0442</td>
</tr>
<tr>
<td>$\pi/3$</td>
<td>$\pi/6$</td>
<td>1.2227</td>
</tr>
<tr>
<td>$\pi/4$</td>
<td></td>
<td>1.0974</td>
</tr>
<tr>
<td>$\pi/3$</td>
<td></td>
<td>1.0479</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td></td>
<td>1.0009</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>$\pi/6$</td>
<td>1.1630</td>
</tr>
<tr>
<td>$\pi/4$</td>
<td></td>
<td>1.0403</td>
</tr>
<tr>
<td>$\pi/3$</td>
<td></td>
<td>0.9932</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td></td>
<td>0.9562</td>
</tr>
</tbody>
</table>
Table 6.8 Frequency parameters $\Omega = \omega a \sqrt{t/E}$ for fully clamped, thin, open spherical shell panels with $a=1\text{m}$, $f_0=\pi/4$ and $t/a=0.01$

<table>
<thead>
<tr>
<th>$L_f$</th>
<th>$L_s$</th>
<th>Mode number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi/6$</td>
<td>$\pi/6$</td>
<td>1.3913</td>
<td>1.4113</td>
<td>1.4617</td>
<td>1.7721</td>
<td>1.8609</td>
<td>2.1428</td>
<td></td>
</tr>
<tr>
<td>$\pi/4$</td>
<td>$\pi/6$</td>
<td>1.2308</td>
<td>1.2733</td>
<td>1.2806</td>
<td>1.4411</td>
<td>1.4899</td>
<td>1.6946</td>
<td></td>
</tr>
<tr>
<td>$\pi/3$</td>
<td>$\pi/6$</td>
<td>1.1875</td>
<td>1.1972</td>
<td>1.2327</td>
<td>1.3380</td>
<td>1.3777</td>
<td>1.3875</td>
<td></td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>$\pi/6$</td>
<td>1.1500</td>
<td>1.1550</td>
<td>1.2023</td>
<td>1.2270</td>
<td>1.2483</td>
<td>1.2600</td>
<td></td>
</tr>
<tr>
<td>$\pi/4$</td>
<td>$\pi/6$</td>
<td>1.2404</td>
<td>1.2580</td>
<td>1.3127</td>
<td>1.4632</td>
<td>1.4704</td>
<td>1.5561</td>
<td></td>
</tr>
<tr>
<td>$\pi/3$</td>
<td>$\pi/6$</td>
<td>1.1196</td>
<td>1.1227</td>
<td>1.1440</td>
<td>1.2218</td>
<td>1.2428</td>
<td>1.3670</td>
<td></td>
</tr>
<tr>
<td>$\pi/4$</td>
<td>$\pi/3$</td>
<td>1.0733</td>
<td>1.0812</td>
<td>1.0874</td>
<td>1.1514</td>
<td>1.1622</td>
<td>1.2305</td>
<td></td>
</tr>
<tr>
<td>$\pi/3$</td>
<td>$\pi/3$</td>
<td>1.0305</td>
<td>1.0527</td>
<td>1.0527</td>
<td>1.0937</td>
<td>1.1008</td>
<td>1.1174</td>
<td></td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>$\pi/3$</td>
<td>1.1849</td>
<td>1.1979</td>
<td>1.2486</td>
<td>1.3314</td>
<td>1.3554</td>
<td>1.3620</td>
<td></td>
</tr>
<tr>
<td>$\pi/3$</td>
<td>$\pi/4$</td>
<td>1.0713</td>
<td>1.0794</td>
<td>1.0902</td>
<td>1.1488</td>
<td>1.1685</td>
<td>1.2099</td>
<td></td>
</tr>
<tr>
<td>$\pi/3$</td>
<td>$\pi/3$</td>
<td>1.0312</td>
<td>1.0378</td>
<td>1.0405</td>
<td>1.0862</td>
<td>1.0933</td>
<td>1.1515</td>
<td></td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>$\pi/2$</td>
<td>0.9878</td>
<td>1.0015</td>
<td>1.0100</td>
<td>1.0387</td>
<td>1.0472</td>
<td>1.0587</td>
<td></td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>$\pi/6$</td>
<td>1.1580</td>
<td>1.1588</td>
<td>1.2195</td>
<td>1.2370</td>
<td>1.2380</td>
<td>1.2925</td>
<td></td>
</tr>
<tr>
<td>$\pi/4$</td>
<td>$\pi/6$</td>
<td>1.0329</td>
<td>1.0559</td>
<td>1.0574</td>
<td>1.0944</td>
<td>1.1021</td>
<td>1.1213</td>
<td></td>
</tr>
<tr>
<td>$\pi/3$</td>
<td>$\pi/2$</td>
<td>0.9878</td>
<td>1.0048</td>
<td>1.0123</td>
<td>1.0409</td>
<td>1.0482</td>
<td>1.0629</td>
<td></td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>$\pi/2$</td>
<td>0.9523</td>
<td>0.9551</td>
<td>0.9556</td>
<td>1.0031</td>
<td>1.0048</td>
<td>1.0143</td>
<td></td>
</tr>
</tbody>
</table>
Table 6.9a Mode shapes comparison with ANSYS for a fully clamped, thin, open cylindrical shell panel with \(a=1\text{m}, \frac{L_s}{a}=1, \frac{L_f}{a} = \frac{\pi}{2}\) and \(t/a=0.01\).

<table>
<thead>
<tr>
<th>Deflection amplitude</th>
<th>Mode shapes</th>
<th>Present</th>
<th>ANSYS</th>
<th>Present</th>
<th>ANSYS</th>
<th>Present</th>
<th>ANSYS</th>
<th>Present</th>
<th>ANSYS</th>
<th>Present</th>
<th>ANSYS</th>
<th>Present</th>
<th>ANSYS</th>
</tr>
</thead>
<tbody>
<tr>
<td>(U_u)</td>
<td>1</td>
<td><img src="image1" alt="Present" /></td>
<td><img src="image2" alt="ANSYS" /></td>
<td>2</td>
<td><img src="image3" alt="Present" /></td>
<td><img src="image4" alt="ANSYS" /></td>
<td>3</td>
<td><img src="image5" alt="Present" /></td>
<td><img src="image6" alt="ANSYS" /></td>
<td>4</td>
<td><img src="image7" alt="Present" /></td>
<td><img src="image8" alt="ANSYS" /></td>
<td>5</td>
</tr>
<tr>
<td>(U_v)</td>
<td><img src="image13" alt="Present" /></td>
<td><img src="image14" alt="ANSYS" /></td>
<td>1</td>
<td><img src="image15" alt="Present" /></td>
<td><img src="image16" alt="ANSYS" /></td>
<td>2</td>
<td><img src="image17" alt="Present" /></td>
<td><img src="image18" alt="ANSYS" /></td>
<td>3</td>
<td><img src="image19" alt="Present" /></td>
<td><img src="image20" alt="ANSYS" /></td>
<td>4</td>
<td><img src="image21" alt="Present" /></td>
</tr>
<tr>
<td>(U_w)</td>
<td><img src="image25" alt="Present" /></td>
<td><img src="image26" alt="ANSYS" /></td>
<td>1</td>
<td><img src="image27" alt="Present" /></td>
<td><img src="image28" alt="ANSYS" /></td>
<td>2</td>
<td><img src="image29" alt="Present" /></td>
<td><img src="image30" alt="ANSYS" /></td>
<td>3</td>
<td><img src="image31" alt="Present" /></td>
<td><img src="image32" alt="ANSYS" /></td>
<td>4</td>
<td><img src="image33" alt="Present" /></td>
</tr>
<tr>
<td>Frequency</td>
<td>Present</td>
<td>0.3458</td>
<td>0.3592</td>
<td>0.4608</td>
<td>0.5133</td>
<td>0.5750</td>
<td>0.5764</td>
<td>ANSYS</td>
<td>0.3462</td>
<td>0.3597</td>
<td>0.4612</td>
<td>0.5135</td>
<td>0.5758</td>
</tr>
</tbody>
</table>
Table 6.9b Mode shape comparison for a fully clamped, thin, open conical shell panel with $L_s=1\text{m}$, $s_0/L_s=1$, $t/L_s=0.01$, $\psi=\pi/6$ and $L_f=\pi/2$.

<table>
<thead>
<tr>
<th>Deflection amplitude</th>
<th>Mode shapes</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td></td>
<td>Present</td>
<td>ANSYS</td>
<td>Present</td>
<td>ANSYS</td>
<td>Present</td>
<td>ANSYS</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.4078</td>
<td>0.4083</td>
<td>0.4360</td>
<td>0.4368</td>
<td>0.6083</td>
<td>0.6090</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.6147</td>
<td>0.6164</td>
<td>0.6808</td>
<td>0.6817</td>
<td>0.6819</td>
<td>0.6833</td>
</tr>
</tbody>
</table>
Table 6.9c Mode shape comparison with ANSYS for a fully clamped, thin, open spherical shell panel with $a=1\text{m}$, $t/a=0.01$, $f_0=\pi/6$, $L_f=\pi/3$ and $L_s=\pi/2$.

<table>
<thead>
<tr>
<th>Deflection amplitude</th>
<th>Mode shapes</th>
<th>(U_u)</th>
<th>(U_v)</th>
<th>(U_w)</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present</td>
<td>Present</td>
<td>ANSYS</td>
<td>Present</td>
<td>ANSYS</td>
<td>Present</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>3</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>4</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>5</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>1.0009</td>
<td>1.0062</td>
<td>1.0179</td>
<td>1.0506</td>
<td>1.0560</td>
</tr>
<tr>
<td></td>
<td>1.0003</td>
<td>1.0056</td>
<td>1.0178</td>
<td>1.0496</td>
<td>1.0559</td>
</tr>
</tbody>
</table>
Table 6.11 Buckling load intensity factors $\lambda = N_c a \sqrt{3(1-v^2)/(Et^2)}$ for thin, open cylindrical shell panels with $a=1$ m, $t/a=0.01$ with CFFF and CCCC boundary conditions

<table>
<thead>
<tr>
<th>$L_s/a$</th>
<th>$L_f$</th>
<th>CFFF Present $p=12$</th>
<th>CCCC Present $p=12$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>$\pi/6$</td>
<td>0.1130</td>
<td>1.2931</td>
</tr>
<tr>
<td></td>
<td>$\pi/4$</td>
<td>0.1121</td>
<td>1.2161</td>
</tr>
<tr>
<td></td>
<td>$\pi/3$</td>
<td>0.1308</td>
<td>1.1919</td>
</tr>
<tr>
<td></td>
<td>$\pi/2$</td>
<td>0.1365</td>
<td>1.1630</td>
</tr>
<tr>
<td>1</td>
<td>$\pi/6$</td>
<td>0.0429</td>
<td>1.1332</td>
</tr>
<tr>
<td></td>
<td>$\pi/4$</td>
<td>0.0605</td>
<td>1.0645</td>
</tr>
<tr>
<td></td>
<td>$\pi/3$</td>
<td>0.0772</td>
<td>1.0528</td>
</tr>
<tr>
<td></td>
<td>$\pi/2$</td>
<td>0.0933</td>
<td>1.0422</td>
</tr>
<tr>
<td>2</td>
<td>$\pi/6$</td>
<td>0.0113</td>
<td>1.2725</td>
</tr>
<tr>
<td></td>
<td>$\pi/4$</td>
<td>0.0450</td>
<td>1.1816</td>
</tr>
<tr>
<td></td>
<td>$\pi/3$</td>
<td>0.0375</td>
<td>1.1154</td>
</tr>
<tr>
<td></td>
<td>$\pi/2$</td>
<td>0.0689</td>
<td>1.0569</td>
</tr>
<tr>
<td>5</td>
<td>$\pi/6$</td>
<td>0.0018</td>
<td>2.7898</td>
</tr>
<tr>
<td></td>
<td>$\pi/4$</td>
<td>0.0085</td>
<td>1.7683</td>
</tr>
<tr>
<td></td>
<td>$\pi/3$</td>
<td>0.0248</td>
<td>1.4430</td>
</tr>
<tr>
<td></td>
<td>$\pi/2$</td>
<td>0.0217</td>
<td>1.2442</td>
</tr>
</tbody>
</table>
Table 6.12 Frequency parameters \( \Omega = \alpha \sqrt{\rho/E} \) for fully clamped, thin, open cylindrical shell panels with \( a=1 \text{m} \), \( L_f = \pi/2 \) or \( \pi \), \( t/a = 0.01 \) or 0.001.

<table>
<thead>
<tr>
<th>( L_s/a )</th>
<th>( L_f )</th>
<th>( t/a )</th>
<th>Mode number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ( \pi/2)</td>
<td>0.01</td>
<td>Present ( p=12 )</td>
<td>0.3458</td>
<td>0.3592</td>
<td>0.4608</td>
<td>0.5133</td>
<td>0.5750</td>
<td>0.5764</td>
<td>0.6707</td>
<td>0.6907</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Present ( p=14 )</td>
<td>0.3458</td>
<td>0.3592</td>
<td>0.4605</td>
<td>0.5123</td>
<td>0.5749</td>
<td>0.5763</td>
<td>0.6664</td>
<td>0.6892</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>ANSYS</td>
<td>0.3462</td>
<td>0.3597</td>
<td>0.4612</td>
<td>0.5135</td>
<td>0.5758</td>
<td>0.5774</td>
<td>0.6670</td>
<td>0.6909</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Relative error %</td>
<td>0.1155</td>
<td>0.1390</td>
<td>0.1518</td>
<td>0.2337</td>
<td>0.1905</td>
<td>0.0900</td>
<td>0.2461</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \pi/2) 0.001</td>
<td>Present ( p=12 )</td>
<td>0.1206</td>
<td>0.1276</td>
<td>0.1627</td>
<td>0.1631</td>
<td>0.2182</td>
<td>0.2186</td>
<td>0.2264</td>
<td>0.2645</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Present ( p=18 )</td>
<td>0.1131</td>
<td>0.1139</td>
<td>0.1295</td>
<td>0.1334</td>
<td>0.1583</td>
<td>0.1645</td>
<td>0.1858</td>
<td>0.1860</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Present ( p=20 )</td>
<td>0.1131</td>
<td>0.1139</td>
<td>0.1293</td>
<td>0.1328</td>
<td>0.1555</td>
<td>0.1613</td>
<td>0.1848</td>
<td>0.1852</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>ANSYS</td>
<td>0.1131</td>
<td>0.1139</td>
<td>0.1293</td>
<td>0.1327</td>
<td>0.1550</td>
<td>0.1607</td>
<td>0.1847</td>
<td>0.1851</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Relative error %</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0754</td>
<td>0.3226</td>
<td>0.3734</td>
<td>0.0541</td>
<td>0.0540</td>
<td></td>
</tr>
<tr>
<td>( \pi) 0.01</td>
<td>Present ( p=12 )</td>
<td>0.3235</td>
<td>0.3326</td>
<td>0.3863</td>
<td>0.3946</td>
<td>0.5025</td>
<td>0.5194</td>
<td>0.574</td>
<td>0.5757</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Present ( p=18 )</td>
<td>0.3152</td>
<td>0.3153</td>
<td>0.3480</td>
<td>0.3493</td>
<td>0.4008</td>
<td>0.4081</td>
<td>0.4782</td>
<td>0.5006</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Present ( p=20 )</td>
<td>0.3152</td>
<td>0.3153</td>
<td>0.3478</td>
<td>0.3491</td>
<td>0.3981</td>
<td>0.4062</td>
<td>0.4653</td>
<td>0.4891</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>ANSYS</td>
<td>0.3156</td>
<td>0.3158</td>
<td>0.3484</td>
<td>0.3496</td>
<td>0.3986</td>
<td>0.4066</td>
<td>0.4638</td>
<td>0.4867</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Relative error %</td>
<td>0.1267</td>
<td>0.1583</td>
<td>0.1722</td>
<td>0.1430</td>
<td>0.1254</td>
<td>0.0984</td>
<td>0.3234</td>
<td>0.4931</td>
<td></td>
</tr>
</tbody>
</table>
### Table 6.12 Frequency parameters $\Omega = \alpha \sqrt{\rho / E}$ for fully clamped, thin, open cylindrical shell panels with $a=1\text{m}$, $L_f=\pi/2$ or $\pi$, $t/a=0.01$ or 0.001 (Cont’d).

<table>
<thead>
<tr>
<th>$L_o/a$</th>
<th>$L_f$</th>
<th>$t/a$</th>
<th>Mode number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\pi$</td>
<td>0.001</td>
<td>Present $p=12$</td>
<td>0.1959</td>
<td>0.2264</td>
<td>0.2676</td>
<td>0.3192</td>
<td>0.3881</td>
<td>0.4013</td>
<td>0.4398</td>
<td>0.4886</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Present $p=18$</td>
<td>0.1260</td>
<td>0.1323</td>
<td>0.1440</td>
<td>0.1514</td>
<td>0.1814</td>
<td>0.1848</td>
<td>0.2185</td>
<td>0.2370</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Present $p=22$</td>
<td>0.1136</td>
<td>0.1152</td>
<td>0.1218</td>
<td>0.1262</td>
<td>0.1428</td>
<td>0.1451</td>
<td>0.1665</td>
<td>0.1737</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Present $p=24$</td>
<td>0.1106</td>
<td>0.1119</td>
<td>0.1180</td>
<td>0.1190</td>
<td>0.1309</td>
<td>0.1343</td>
<td>0.1512</td>
<td>0.1619</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Present $p=28$</td>
<td>0.1089</td>
<td>0.1091</td>
<td>0.1138</td>
<td>0.1140</td>
<td>0.1221</td>
<td>0.1241</td>
<td>0.1368</td>
<td>0.1369</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>ANSYS</td>
<td>0.1090</td>
<td>0.1090</td>
<td>0.1133</td>
<td>0.1134</td>
<td>0.1203</td>
<td>0.1206</td>
<td>0.1296</td>
<td>0.1306</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Relative error %</td>
<td>0.0917</td>
<td>0.0917</td>
<td>0.4413</td>
<td>0.5291</td>
<td>1.4963</td>
<td>2.9022</td>
<td>5.5556</td>
<td>4.8239</td>
</tr>
</tbody>
</table>

### Table 6.13 Frequency parameters $\Omega = \alpha \sqrt{\rho / E}$ for fully clamped, thin, open conical shell panels with $L_o=1\text{m}$, $\psi=\pi/4$, $s_0/(L_o+s_0)=0.4$, $L_f=\pi/2$ or $\pi$, $t/L_o=0.01$ or 0.001.

<table>
<thead>
<tr>
<th>$s_0/(L_o+s_0)$</th>
<th>$L_f$</th>
<th>$t/L_o$</th>
<th>Mode number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>$\pi/2$</td>
<td>0.01</td>
<td>Present $p=12$</td>
<td>0.3471</td>
<td>0.3680</td>
<td>0.5001</td>
<td>0.5107</td>
<td>0.5747</td>
<td>0.5855</td>
<td>0.6736</td>
<td>0.7248</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>ANSYS</td>
<td>0.3474</td>
<td>0.3685</td>
<td>0.5006</td>
<td>0.5119</td>
<td>0.5753</td>
<td>0.5865</td>
<td>0.6742</td>
<td>0.7247</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Relative error %</td>
<td>0.0864</td>
<td>0.1357</td>
<td>0.0999</td>
<td>0.2344</td>
<td>0.1043</td>
<td>0.1705</td>
<td>0.0890</td>
<td>0.0138</td>
</tr>
</tbody>
</table>
Table 6.13 Frequency parameters $\Omega = \alpha \sqrt{\rho / E}$ for fully clamped, thin, open conical shell panels with $L_s=1m$, $\psi=\pi/4$, $s_0/(L_s+s_0)=0.4$, $L_f=\pi/2$ or $\pi$, $t/L_s=0.01$ or 0.001. (Cont’d)

<table>
<thead>
<tr>
<th>$s_0/(L_s+s_0)$</th>
<th>$L_f$</th>
<th>$t/L_s$</th>
<th>Mode number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>$\pi/2$</td>
<td>0.001</td>
<td>Present $p=12$</td>
<td>0.1144</td>
<td>0.1145</td>
<td>0.1410</td>
<td>0.1556</td>
<td>0.1926</td>
<td>0.1955</td>
<td>0.2093</td>
<td>0.2120</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Present $p=18$</td>
<td>0.1120</td>
<td>0.1131</td>
<td>0.1330</td>
<td>0.1332</td>
<td>0.1563</td>
<td>0.1640</td>
<td>0.1865</td>
<td>0.1870</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Present $p=20$</td>
<td>0.1120</td>
<td>0.1131</td>
<td>0.1329</td>
<td>0.1332</td>
<td>0.1560</td>
<td>0.1629</td>
<td>0.1855</td>
<td>0.1861</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>ANSYS</td>
<td>0.1120</td>
<td>0.1131</td>
<td>0.1329</td>
<td>0.1331</td>
<td>0.1559</td>
<td>0.1627</td>
<td>0.1851</td>
<td>0.1857</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Relative error %</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0751</td>
<td>0.0641</td>
<td>0.1229</td>
<td>0.2161</td>
<td>0.2154</td>
</tr>
<tr>
<td>$\pi$</td>
<td>0.01</td>
<td></td>
<td>Present $p=12$</td>
<td>0.3058</td>
<td>0.3078</td>
<td>0.3551</td>
<td>0.3699</td>
<td>0.4490</td>
<td>0.4883</td>
<td>0.5285</td>
<td>0.5302</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Present $p=16$</td>
<td>0.3041</td>
<td>0.3064</td>
<td>0.3449</td>
<td>0.3543</td>
<td>0.4096</td>
<td>0.4218</td>
<td>0.5047</td>
<td>0.5136</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Present $p=18$</td>
<td>0.3041</td>
<td>0.3064</td>
<td>0.3448</td>
<td>0.3541</td>
<td>0.4084</td>
<td>0.4185</td>
<td>0.4899</td>
<td>0.5046</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>ANSYS</td>
<td>0.3044</td>
<td>0.3068</td>
<td>0.3453</td>
<td>0.3546</td>
<td>0.4088</td>
<td>0.4190</td>
<td>0.4885</td>
<td>0.5025</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Relative error %</td>
<td>0.0986</td>
<td>0.1304</td>
<td>0.1448</td>
<td>0.1410</td>
<td>0.0978</td>
<td>0.1193</td>
<td>0.2866</td>
<td>0.4179</td>
</tr>
<tr>
<td>$\pi$</td>
<td>0.001</td>
<td></td>
<td>Present $p=12$</td>
<td>0.1441</td>
<td>0.1607</td>
<td>0.1904</td>
<td>0.2299</td>
<td>0.2762</td>
<td>0.2961</td>
<td>0.3114</td>
<td>0.3602</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Present $p=18$</td>
<td>0.1098</td>
<td>0.1122</td>
<td>0.1224</td>
<td>0.1243</td>
<td>0.1427</td>
<td>0.1554</td>
<td>0.1862</td>
<td>0.1941</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Present $p=24$</td>
<td>0.1060</td>
<td>0.1061</td>
<td>0.1116</td>
<td>0.1125</td>
<td>0.1223</td>
<td>0.1223</td>
<td>0.1365</td>
<td>0.1405</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Present $p=28$</td>
<td>0.1059</td>
<td>0.1061</td>
<td>0.1114</td>
<td>0.1119</td>
<td>0.1200</td>
<td>0.1203</td>
<td>0.1310</td>
<td>0.1316</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>ANSYS</td>
<td>0.1060</td>
<td>0.1061</td>
<td>0.1115</td>
<td>0.1120</td>
<td>0.1200</td>
<td>0.1204</td>
<td>0.1301</td>
<td>0.1312</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Relative error %</td>
<td>0.0943</td>
<td>0.0000</td>
<td>0.0897</td>
<td>0.0893</td>
<td>0.0000</td>
<td>0.0831</td>
<td>0.6918</td>
<td>0.3049</td>
</tr>
</tbody>
</table>
Table 6.14 Frequency parameters $\Omega=\alpha\sqrt{\rho/E}$ for fully clamped, thin, open spherical shell panels with $a=1m$, $f_0=\pi/4$, $L_f=\pi/4$, $L_s=\pi/2$ or $\pi$, $t/a=0.01$ or 0.001

<table>
<thead>
<tr>
<th>$L_f$</th>
<th>$L_s$</th>
<th>$t/a$</th>
<th>Mode number</th>
<th>Present $p=12$</th>
<th>Present $p=18$</th>
<th>ANSYS</th>
<th>Relative error %</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi/4$</td>
<td>$\pi/2$</td>
<td>0.01</td>
<td>1</td>
<td>1.0305</td>
<td>1.0305</td>
<td>1.0300</td>
<td>0.0485</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2</td>
<td>1.0527</td>
<td>1.0526</td>
<td>1.0526</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>3</td>
<td>1.0937</td>
<td>1.0938</td>
<td>1.0927</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>4</td>
<td>1.1008</td>
<td>1.1005</td>
<td>1.1026</td>
<td>0.0091</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>5</td>
<td>1.1174</td>
<td>1.1173</td>
<td>1.1173</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>6</td>
<td>1.1318</td>
<td>1.1318</td>
<td>1.1325</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>7</td>
<td>1.1528</td>
<td>1.1525</td>
<td>1.1523</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.001</td>
<td>1</td>
<td>0.9826</td>
<td>0.9826</td>
<td>0.9806</td>
<td>0.2040</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2</td>
<td>0.9835</td>
<td>0.9835</td>
<td>0.9824</td>
<td>0.1120</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>3</td>
<td>0.9844</td>
<td>0.9844</td>
<td>0.9832</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>4</td>
<td>0.9888</td>
<td>0.9888</td>
<td>0.9877</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>5</td>
<td>0.9927</td>
<td>0.9927</td>
<td>0.9913</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>6</td>
<td>0.9950</td>
<td>0.9950</td>
<td>0.9924</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>7</td>
<td>0.9963</td>
<td>0.9962</td>
<td>0.9939</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>8</td>
<td>0.9966</td>
<td>0.9966</td>
<td>0.9947</td>
<td>0.0000</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td></td>
<td>0.01</td>
<td>1</td>
<td>0.9964</td>
<td>0.9964</td>
<td>0.9964</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2</td>
<td>1.0279</td>
<td>1.0279</td>
<td>1.0276</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>3</td>
<td>1.0386</td>
<td>1.0378</td>
<td>1.0364</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>4</td>
<td>1.0388</td>
<td>1.0379</td>
<td>1.0366</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>5</td>
<td>1.0544</td>
<td>1.0499</td>
<td>1.0479</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>6</td>
<td>1.0581</td>
<td>1.0506</td>
<td>1.0491</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>7</td>
<td>1.0611</td>
<td>1.0610</td>
<td>1.0612</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>8</td>
<td>1.0852</td>
<td>1.0695</td>
<td>1.0668</td>
<td>0.0000</td>
</tr>
<tr>
<td>$\pi$</td>
<td></td>
<td>0.01</td>
<td>1</td>
<td>0.9428</td>
<td>0.9427</td>
<td>0.9413</td>
<td>0.1487</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2</td>
<td>0.9816</td>
<td>0.9816</td>
<td>0.9763</td>
<td>0.5429</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>3</td>
<td>0.9817</td>
<td>0.9817</td>
<td>0.9763</td>
<td>0.5531</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>4</td>
<td>0.9828</td>
<td>0.9825</td>
<td>0.9764</td>
<td>0.6247</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>5</td>
<td>0.9831</td>
<td>0.9829</td>
<td>0.9764</td>
<td>0.6657</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>6</td>
<td>0.9845</td>
<td>0.9831</td>
<td>0.9780</td>
<td>0.5215</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>7</td>
<td>0.9864</td>
<td>0.9844</td>
<td>0.9780</td>
<td>0.6544</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>8</td>
<td>0.9863</td>
<td>0.9844</td>
<td>0.9785</td>
<td>0.7971</td>
</tr>
</tbody>
</table>
6.7 Conclusion

Natural frequencies and vibration mode shapes of open, thin, cylindrical, conical and spherical shell panels are obtained by polynomial $C^0$ $p$-elements and compared well with the existed results. Buckling loads of axial compression cylindrical shell panels are also studied. The frequencies are decreased with the circumferential angle and aspect ratio, separately. However, the buckling loads of cylindrical shell panels are not changing monotonically with the circumferential angle and the aspect ratio. Moreover, the polynomial $p$-elements can also be extended to solve shell panels with deeper circumferential angle and thinner thickness. However, much more polynomial terms will be needed to achieve accurate results.
CHAPTER 7

CONCLUSION AND RECOMMENDATIONS

7.1 Conclusion and Limitations of the Research

In the research presented in this thesis, Fourier $p$-elements and polynomial $p$-elements using trigonometric functions and Legendre orthogonal polynomials, respectively, as the shape functions have been employed to solve the vibration, buckling and dynamic stability problems of plane beam columns, space beams, pre-twisted straight beams, Mindlin plates and open shell panels. The accuracy of the $p$-element method has been confirmed through comparison with the solutions obtained by the dynamic stiffness method, the numerical results obtained by the conventional FEM program, ANSYS, and/or by other convergent solution methods.

The main contribution of this thesis lies in its extension of the $p$-version element to the solution of a number of special structures using a newly established theory. Chapter 2 reports original research on the buckling of plane beam columns subject to follower concentrated or uniformly distributed tensile loads. In Chapter 3, the unequally shared end torque theory introduced by Leung is adopted, and the axial-torsional buckling of straight space beams is investigated. In Chapter 4, an
original theory of pre-twisted straight beams is established, and it is found that a pre-twisted beam will buckle not only in the major-axis, but also in the minor-axis, when subject to end moment or end shear loads. Chapter 5 presents novel research on the buckling problems of trapezoidal plates investigated by $p$-version elements that can’t be solved by the Ritz method because of mapping difficulties. The vibration and buckling problems of plate systems composed of different plate element thicknesses are also investigated. Finally, in Chapter 6, the theories of open, thin shell panels, including cylindrical, conical and spherical shell panels, are established, and the polynomial $p$-elements are extended to solve the vibration problems of open shell panels. The results show that when shell panels become deeper or thinner, the vibration mode numbers in their circumferential direction increases significantly, and many more additional polynomial terms are needed to achieve accurate frequencies.

It has been confirmed that the proposed $p$-version elements with analytical integration are very useful for the vibration and buckling analysis of structures. With an increasing number of additional terms in the shape functions, the stiffness of the $p$-elements decreases, and thus the downward convergence of these elements is observed. This study proves that the convergence of the proposed $p$-elements is rapid with respect to the increasing number of additional Fourier or polynomial terms in the shape functions, and their solutions are much more accurate than those of the $h$-version elements for the same number of DOFs. Only one element is needed in $p$-element studies. However, in the formulation and operation of the element matrices, the $p$-version element is more difficult to apply than the $h$-version element, because the order and bandwidth of the matrices of the former are larger than those of the latter. To obtain the analytical integration of the element matrices, much more computational time and space must be spent on the
symbol integration procedure, especially for structures subject to complicated external loads. It is also difficult to extend the $p$-version element to structures with complicated shapes in mapping problems. When solving Mindlin plate buckling problems, because the stress distribution in the structures is not assumed to be uniform and it is difficult to find a suitable overall function that fits the stress distribution inside the plate, it is very difficult to obtain the analytical integration of the geometric stiffness matrices using the $p$-version element method. Numerical integration procedures should be used instead, although the results will be less accurate.

### 7.2 Recommendations for Further Research

The following areas are recommended for the further extension and improvement of the $p$-version element method proposed in this thesis.

1. By drawing on the results of the study carried out on plane, space and pre-twisted straight beams, it would be very easy to extend the $p$-version method to solve the vibration and buckling problems of beams with a pre-twist ratio, variable curvature and tortuosity, and a variable cross-section.

2. On the basis of the research carried out on cylindrical, conical and spherical shell panels, the $p$-version method could also be easily extended to solve the problems of other shell panels, such as rotational symmetric shell panels and toroidal shell panels.

3. The $p$-version element method developed herein offers error estimates that are more precise than those of the $h$-version method, and they can be calculated both locally and globally which allows highly accurate solutions to be obtained at a point, such as in fracture or fatigue assessment.
When solving crack problems, the $p$-version element can be employed as a high-precision element in combination with the fractal two-level finite element proposed by Leung [207-209] or the infinite element method introduced by Ying [210-213], in which only one-layer element matrices around the crack tip are ultimately used. The stress intensity factors and T-stress are primarily affected by the accuracy of the element matrices in the first layer.

4. In the study of Mindlin plate vibration and buckling problems reported in Chapter 5, the triangular or polygonal plates were divided into trapezoidal and rectangular plate elements, but the expected analytical equal natural frequencies or buckling loads differed from each other. To overcome this problem, finding a triangular $p$-element with good numerical performance is essential. This triangular $p$-element could also be extended to structures with complicated boundary shapes.

5. In the investigation of the vibration problems of thinner, deeper shell panels reported in Chapter 6, it was found that many more Legendre polynomial terms would be needed to achieve accurate results. When employing high-order polynomial $p$-elements, the numerical error problems become severe, because the numerical effective digit of the results decreases rapidly with high-order polynomial terms. To reduce the numerical error, the symbolic toolbox in MATLAB was widely used in the integration procedures reported in this thesis, with the intermediate integration values kept as symbolic values until the integration procedure was complete. In future research, explicit expressions of various kinds of integration could be deduced in advance, and an expert in symbolic operation software could be asked to guarantee the accuracy of the integration results.
BIBLIOGRAPHY


[39] N.S. Bardell, The application of symbolic computing to the hierarchical


1980.


[116] Q.S. Li, Stability of non-uniform columns under the combined action of


[134] A.Y.T. Leung, Non-conservative dynamic axial-torsional buckling of


[146] B. Dawson, Coupled bending vibrations of pre-twisted cantilever blading


[156] E. Petrov, M. Geradin, Finite element theory for curved and twisted beams


[168] ANSYS Inc., 201 Johnson Road, Houston, PA 15342-1300, U.S.A.


[185] C.W. Lim, K.M. Liew, S. Kitipornchai, Vibration of open cylindrical shells:


[204] K.M. Liew, C.W. Lim, A Ritz vibration analysis of doubly curved rectangular shallow shells using a refined first-order theory, *Computer*


APPENDIX 1: SHAPE FUNCTIONS

C$^0$ hierarchical shape functions

\[ f_1 = \frac{(1 - \xi)}{2}, \]
\[ f_2 = \frac{(1 + \xi)}{2}, \]
\[ f_3 = \frac{\xi^2 - 1}{2}, \]
\[ f_4 = \frac{\xi^3 - \xi}{2}, \]
\[ f_5 = \frac{5\xi^4 - 6\xi^2 + 1}{8}, \]
\[ f_6 = \frac{7\xi^5 - 10\xi^3 + 3\xi}{8}, \]
\[ f_7 = \frac{63\xi^6 - 105\xi^4 + 45\xi^2 - 3}{48}, \]
\[ f_8 = \frac{99\xi^7 - 189\xi^5 + 105\xi^3 - 15\xi}{48}, \]
\[ f_9 = \frac{429\xi^8 - 924\xi^6 + 630\xi^4 - 140\xi^2 + 5}{128}, \]
\[ f_{10} = \frac{715\xi^9 - 1716\xi^7 + 1386\xi^5 - 420\xi^3 + 35\xi}{128}, \]
\[ f_{11} = \frac{2431\xi^{10} - 6435\xi^8 + 6006\xi^6 - 2310\xi^4 + 315\xi^2 - 7}{256}, \]
\[ f_{12} = \frac{4199\xi^{11} - 12155\xi^9 + 12870\xi^7 - 6006\xi^5 + 1155\xi^3 - 63\xi}{256}, \]
\[ f_{13} = \frac{29393\xi^{12} - 92378\xi^{10} + 109395\xi^8 - 60060\xi^6 + 15015\xi^4 - 1386\xi^2 + 21}{1024}, \]
\[ f_{14} = \frac{52003\xi^{13} - 176358\xi^{11} + 230945\xi^9 - 145860\xi^7 + 45045\xi^5 - 6006\xi^3 + 231\xi}{1024}, \]

where \(-1 < \xi < 1\).
By mapping $\xi$ into $(0, 1)$, the hierarchical shape functions become:

\begin{align*}
\overline{f}_1 &= 1 - \xi, \\
\overline{f}_2 &= \xi, \\
\overline{f}_3 &= 2\xi^2 - 2\xi, \\
\overline{f}_4 &= 4\xi^3 - 6\xi^2 + 2\xi, \\
\overline{f}_5 &= 10\xi^4 - 20\xi^3 + 12\xi^2 - 2\xi, \\
\overline{f}_6 &= 28\xi^5 - 70\xi^4 + 60\xi^3 - 20\xi^2 + 2\xi, \\
\overline{f}_7 &= 84\xi^6 - 252\xi^5 + 280\xi^4 - 140\xi^3 + 30\xi^2 - 2\xi, \\
\overline{f}_8 &= 264\xi^7 - 924\xi^6 + 1260\xi^5 - 840\xi^4 + 280\xi^3 - 42\xi^2 + 2\xi, \\
\overline{f}_9 &= 858\xi^8 - 3432\xi^7 + 5544\xi^6 - 4620\xi^5 + 2100\xi^4 - 504\xi^3 + 56\xi^2 - 2\xi, \\
\overline{f}_{10} &= 2860\xi^9 - 12870\xi^8 + 24024\xi^7 - 24024\xi^6 + 13860\xi^5 - 4620\xi^4 + 840\xi^3 \\
& \quad - 72\xi^2 + 2\xi, \\
\overline{f}_{11} &= 9724\xi^{10} - 48620\xi^9 + 102960\xi^8 - 120120\xi^7 + 84084\xi^6 - 36036\xi^5 + 9240\xi^4 \\
& \quad - 1320\xi^3 + 90\xi^2 - 2\xi, \\
\overline{f}_{12} &= 33592\xi^{11} - 184756\xi^{10} + 437580\xi^9 - 583440\xi^8 + 480480\xi^7 - 252252\xi^6 \\
& \quad + 84084\xi^5 - 17160\xi^4 + 1980\xi^3 - 110\xi^2 + 2\xi, \\
\overline{f}_{13} &= 117572\xi^{12} - 705432\xi^{11} + 1847560\xi^{10} - 2771340\xi^9 + 2625480\xi^8 - 1633632\xi^7 \\
& \quad + 672672\xi^6 - 180180\xi^5 + 30030\xi^4 - 2860\xi^3 + 132\xi^2 - 2\xi, \\
\overline{f}_{14} &= 416024\xi^{13} - 2704156\xi^{12} + 7759752\xi^{11} - 12932920\xi^{10} + 13856700\xi^9 - 9976824\xi^8 \\
& \quad + 4900896\xi^7 - 1633632\xi^6 + 360360\xi^5 - 50050\xi^4 + 4004\xi^3 - 156\xi^2 + 2\xi.
\end{align*}
C⁰ Fourier sine shape functions:

\[ f_i = 1 - \xi, \]
\[ f_2 = \xi, \]
\[ f_{i+2} = \sin \left( i\pi \xi \right), \]

where \( 0 < \xi < 1 \) and \( i = 1, 2, \ldots \).

C¹ Fourier sine shape functions:

\[ f_1 = 1 - 3\xi^2 + 2\xi^3, \]
\[ g_1 = \xi \left( 1 - 2\xi + \xi^2 \right) l, \]
\[ f_2 = 3\xi^2 - 2\xi^3, \]
\[ g_2 = \xi \left( \xi^2 - \xi \right) l, \]
\[ b_i = \left( \xi - \xi^2 \right) \sin i\pi \xi, i = 1, 2, \ldots, (p), \]
\[ \xi = \frac{x}{l} \left( 0 < x < l \right), \]

where \( l \) is the length of a beam and \( 0 < \xi < 1 \) and \( i = 1, 2, \ldots \).
APPENDIX 2: MATRIX ENTRIES FOR COLUMNS

The entries of the stiffness matrix of a uniform beam are

\[
\mathbf{K}^e = \frac{EI}{l^3} \begin{bmatrix}
12 & 6l & -12 & 6l & 0 \\
6l & 4l^2 & -6l & 2l^2 & 0 \\
-12 & -6l & 12 & -6l & 0 \\
6l & 2l^2 & -6l & 4l^2 & 0 \\
0 & 0 & 0 & 0 & \mathbf{k}
\end{bmatrix},
\]

(A2.1)

where the entries of the submatrix \( \mathbf{k} \) are given by

\[
k_{ij} = \begin{cases}
-48i^3 j^3 \left( i^2 + j^2 \right) \left[ 1 + (-1)^{i+j} \right] / \left( i^2 - j^2 \right)^4 & i \neq j, \\
45 + 60 j^2 \pi^2 + j^4 \pi^4 / 60 & i = j.
\end{cases}
\]

The entries of the mass matrix of a uniform beam are

\[
\mathbf{M}^e = \frac{DAI}{420} \begin{bmatrix}
156 & 22l & 54 & -13l & \mathbf{m}_{15} \\
22l & 4l^2 & 13l & -3l^2 & \mathbf{m}_{25} \\
54 & 13l & 156 & -22l & \mathbf{m}_{35} \\
-13l & -3l^2 & -22l & 4l^2 & \mathbf{m}_{45} \\
\mathbf{m}_{51} & \mathbf{m}_{52} & \mathbf{m}_{53} & \mathbf{m}_{54} & \mathbf{m}_{55}
\end{bmatrix},
\]

(A2.2)

where

\[
\mathbf{m}_{15} = \frac{\mathbf{m}_{15}}{420} = \frac{2}{j^2 \pi^5} \left[ 60 + j^2 \pi^2 + 60 (-1)^j \right],
\]

\[
\mathbf{m}_{25} = \frac{\mathbf{m}_{25}}{420} = \frac{1}{j^2 \pi^5} \left[ 72 - 2 j^2 \pi^2 + 48 (-1)^j \right] l,
\]

\[
\mathbf{m}_{35} = \frac{\mathbf{m}_{35}}{420} = \frac{-1}{j^2 \pi^5} \left[ 120 + (120 + 2 j^2 \pi^2) (-1)^j \right],
\]

\[
\mathbf{m}_{45} = \frac{\mathbf{m}_{45}}{420} = \frac{2}{j^2 \pi^5} \left[ 24 + (36 - j^2 \pi^2) (-1)^j \right] l,
\]
The entries of the geometric matrix of a beam subject to uniform axial force are

\[
\mathbf{m}_{55} = \frac{420}{(i^2 - j^2)^4 \pi^4} \begin{cases} 
-48ij(i^2 + j^2) \left[ 1 + (-1)^{i+j} \right] & i \neq j, \\
45 + j^4 \pi^4 & i = j.
\end{cases}
\]

The entries of the geometric matrix of a beam subject to uniform axial force are

\[
\mathbf{G}^e = \frac{1}{l} \begin{bmatrix}
6 & \frac{1}{10}l & -6 & \frac{1}{10}l & g_{s15} \\
\frac{1}{10}l & \frac{2}{15}l^2 & -\frac{1}{10}l & -\frac{1}{30}l^2 & g_{s25} \\
-\frac{6}{5} & -\frac{1}{10}l & 6 & \frac{1}{10} & g_{s35} \\
\frac{1}{10}l & -\frac{1}{30}l^2 & -\frac{1}{10}l & \frac{2}{15}l^2 & g_{s45} \\
g_{s51} & g_{s52} & g_{s53} & g_{s54} & g_{s55}
\end{bmatrix},
\]

where

\[
g_{s15} = g_{s51} = \frac{36}{j^3 \pi^3} \left[ 1 + (-1)^j \right],
\]

\[
g_{s25} = g_{s52} = \frac{4}{j^3 \pi^3} \left[ 5 + 4(-1)^j \right] l,
\]

\[
g_{s35} = g_{s53} = -\frac{36}{j^3 \pi^3} \left[ 1 + (-1)^j \right],
\]

\[
g_{s45} = g_{s54} = \frac{4}{j^3 \pi^3} \left[ 4 + 5(-1)^j \right] l,
\]

\[
g_{s55} = \begin{cases} 
-16ij(i^4 + 4i^2j^2 + j^4) \left[ 1 + (-1)^{i+j} \right] & i \neq j, \\
-30 + 20j^2 \pi^2 + 2j^4 \pi^4 & i = j,
\end{cases}
\]

and \( i, j = 1, 2, 3 \ldots \).
The entries of \( \mathbf{G}^e \) and \( \bar{\mathbf{G}}^e \) of a beam subject to uniform distributed loads are:

\[
\mathbf{G}^e = \frac{1}{l} \begin{bmatrix}
\frac{3}{5} & 0 & -\frac{3}{5} & \frac{1}{10} & \mathbf{g}_{51} \\
0 & \frac{1}{10} & 0 & -\frac{1}{60} & \mathbf{g}_{25} \\
-\frac{3}{5} & 0 & \frac{3}{5} & -\frac{1}{10} & \mathbf{g}_{53} \\
\frac{1}{10} & -\frac{1}{60} & -\frac{1}{10} & \frac{1}{30} & \mathbf{g}_{45}
\end{bmatrix}, \quad (A2.4)
\]

where

\[
\mathbf{g}_{15} = \mathbf{g}_{51}^T = \frac{-12 \left[ 36 - 5 j^2 \pi^2 + 2(-18 + j^2 \pi^2)(-1)^j \right]}{j^5 \pi^5},
\]

\[
\mathbf{g}_{25} = \mathbf{g}_{52}^T = \frac{-216 + 38 j^2 \pi^2 - 8(-27 + j^2 \pi^2)(-1)^j}{j^5 \pi^5} l,
\]

\[
\mathbf{g}_{35} = \mathbf{g}_{53}^T = \frac{12 \left[ 36 - 5 j^2 \pi^2 + 2(-18 + j^2 \pi^2)(-1)^j \right]}{j^5 \pi^5},
\]

\[
\mathbf{g}_{45} = \mathbf{g}_{54}^T = \frac{-24(-9 + j^2 \pi^2) - 18(-12 + j^2 \pi^2)(-1)^j}{j^5 \pi^5} l,
\]

\[
\mathbf{g}_{55} = 8ij \begin{cases}
\frac{-3 \left( i^8 \pi^2 + j^6 \left(-9 + j^2 \pi^2\right) + i^2 j^4 \left(-71 + 2 j^2 \pi^2\right) \right)}{(i^2 - j^2)^3 \pi^4} & i \neq j, \\
\frac{-15 + 10 j^2 \pi^2 + j^4 \pi^4}{120 j^2 \pi^2} & i = j,
\end{cases}
\]
and

\[
\begin{bmatrix}
\frac{1}{2} & \frac{1}{10} & \frac{1}{2} & \frac{1}{10} & -
g_{15} \\
-\frac{1}{10} & 0 & \frac{1}{10} & \frac{1}{60} & -
g_{25} \\
\frac{1}{2} & \frac{1}{10} & \frac{1}{2} & \frac{1}{10} & -
g_{35} \\
\frac{1}{10} & \frac{1}{60} & -\frac{1}{10} & 0 & -
g_{45} \\
-g_{51} & -g_{52} & -g_{53} & -g_{54} & -g_{55}
\end{bmatrix}
\]

where

\[
\begin{align*}
g_{15} &= -\frac{12 \left( -12 + j^2 \pi^2 \right) \left( -1 + (-1)^j \right)}{j^5 \pi^5}, \\
g_{25} &= -\frac{72 - 10 j^2 \pi^2 + 4 \left( -18 + j^2 \pi^2 \right) (-1)^j}{j^5 \pi^5} l, \\
g_{35} &= -\frac{12 \left( -12 + j^2 \pi^2 \right) \left( -1 + (-1)^j \right)}{j^5 \pi^5}, \\
g_{45} &= -\frac{72 - 4 j^2 \pi^2 + 2 \left( -36 + 5 j^2 \pi^2 \right) (-1)^j}{j^5 \pi^5} l, \\
g_{55} &= \begin{cases}
4ij \left[ -1 + (-1)^j \right] \\
\frac{i^6 \pi^2 + j^4 \left( -18 + j^2 \pi^2 \right)}{-i^4 \left( 18 + j^2 \pi^2 \right) - j^2 \pi^2 \left( 60 + j^2 \pi^2 \right)} \\
\left( i^2 - j^2 \right)^{\frac{1}{5}} \pi^3 
\end{cases}
\end{align*}
\]

and \( i, j = 1, 2, 3, \ldots, p \).
The coefficients of the stiffness matrix of the tapered beam with the dimension given in Eq. (2.20) are:

\[
\mathbf{K}^e = \frac{EI_0}{l^4} \begin{bmatrix}
69741 & 4152 & -69741 & 1461 & k_{15} \\
5000 & 625 & 5000 & 200 & \\
4152 & 10791 & -4152 & 5817 & l^2 & k_{25} \\
625 & 2500 & 625 & 2500 & \\
69741 & 4152 & 69741 & 1461 & l & k_{35} \\
5000 & 625 & 5000 & 200 & \\
1461 & 5817 & -1461 & 24891 & l^2 & k_{45} \\
200 & 2500 & 200 & 5000 & \\
k_{51} & k_{52} & k_{53} & k_{54} & k_{55}
\end{bmatrix}
\]

where

\[
k_{15} = k_{51}^T = \frac{-3 \left[576 - 1572 j^2 \pi^2 + 192 (-1)^j \left(-3 + 20 j^2 \pi^2\right)\right]}{500 j^5 \pi^4},
\]

\[
k_{25} = k_{52}^T = -\frac{288 \left(3 - 8 j^2 \pi^2\right) + 72 (-1)^j \left(-12 + 79 j^2 \pi^2\right)}{500 j^5 \pi^4} l,
\]

\[
k_{35} = k_{53}^T = \frac{3 \left[576 - 1572 j^2 \pi^2 + 192 (-1)^j \left(-3 + 20 j^2 \pi^2\right)\right]}{500 j^5 \pi^4},
\]

\[
k_{45} = k_{54}^T = \frac{36 \left(-24 + 67 j^2 \pi^2\right) - 216 (-1)^j \left(-4 + 27 j^2 \pi^2\right)}{500 j^5 \pi^4}.
\]
The mass matrix entries of the tapered beam as shown in Eq. (2.20) are

\[
\mathbf{m}_{ij} = \begin{cases} 
\frac{1}{500(i^2 - j^2)^8 \pi^4} & i \neq j, \\
\begin{pmatrix}
6j^{10} \left( 12 - 79j^2\pi^2 \right) \\
+i^{10}j^6 \left( -474 + 3025j^2\pi^2 \right) \\
+i^{10}j^6 \left( 72 - 13760j^2\pi^2 - 9075j^4\pi^4 \right) \\
-12(-1)^{i+j}ij \\
i^{10}j^4 \left( 10944 - 9494j^2\pi^2 - 9075j^4\pi^4 \right) \\
+i^{10}j^6 \left( 5472 - 23728j^2\pi^2 + 3025j^4\pi^4 \right) \\
+2i^{10}j^4 \left( 4200 - 13760j^2\pi^2 + 3025j^4\pi^4 \right) \\
+i^{10}j^6 \left( 2424 - 9494j^2\pi^2 + 6050j^4\pi^4 \right)
\end{pmatrix} & i = j.
\end{cases}
\]

The mass matrix entries of the tapered beam as shown in Eq. (2.20) are

\[
\begin{bmatrix}
19 & 227 & 27 & -17 & \mathbf{m}_{15} \\
50 & 4200 & 200 & 525 & \mathbf{m}_{51} \\
227 & 83 & 137 & -3 & \mathbf{m}_{25} \\
4200 & 4300 & 1200 & 400 & \mathbf{m}_{55} \\
27 & 137 & 2 & 47 & \mathbf{m}_{35} \\
200 & 4200 & 5 & 840 & \mathbf{m}_{33} \\
17 & 3 & 47 & 17 & \mathbf{m}_{45} \\
525 & 400 & 840 & 1680 & \mathbf{m}_{54} \\
\mathbf{m}_{15} & \mathbf{m}_{52} & \mathbf{m}_{32} & \mathbf{m}_{34} & \mathbf{m}_{55}
\end{bmatrix}
\]

\text{where}

\[
\mathbf{m}_{15} = \mathbf{m}_{51}^T = \frac{720 + 564j^2\pi^2 + 9j^4\pi^4 + 24(-1)^j(-30 + 29j^2\pi^2)}{5j^2\pi^2},
\]
\[
\begin{align*}
\mathbf{m}_{25} &= \mathbf{m}_{52}^T = \frac{360 + 324j^2\pi^2 - 10j^4\pi^4 + 12(-1)^j(-30 + 23j^2\pi^2)}{5j^j\pi^j}, \\
\mathbf{m}_{35} &= \mathbf{m}_{53}^T = -\frac{12(60 + 47j^2\pi^2) + 12(-1)^j(-60 + 58j^2\pi^2 + j^4\pi^4)}{5j^j\pi^j}, \\
\mathbf{m}_{45} &= \mathbf{m}_{54}^T = \frac{6(60 + 38j^2\pi^2) + (-1)^j(-360 + 432j^2\pi^2 - 11j^4\pi^4)}{5j^j\pi^j}.
\end{align*}
\]

\[
\mathbf{m}_{55} = \begin{cases} 
\begin{bmatrix}
19i^6\pi^2 + i^4(30 - 19j^2\pi^2) \\
+ i^4(30 + 19j^2\pi^2) + i^2(100j^2 - 19j^4\pi^2)
\end{bmatrix}
& \text{for } i \neq j, \\
-12ij & \text{for } i = j,
\end{cases}
\]

and \( i, j = 1, 2, 3, \ldots, p \).
APPENDIX 3: MATRIX ENTRIES FOR SPACE BEAMS

The entries of the stiffness matrix are

\[
K^e = \frac{1}{l^3} \begin{bmatrix}
    k_{11} & k_{12} & k_{13} \\
    k_{21} & k_{22} & k_{23} \\
    k_{31} & k_{32} & k_{33}
\end{bmatrix},
\]

(A3.1)

where

\[
k_{11} (1,1) = 12C_y, k_{11} (1,4) = 6C_y l, k_{11} (2,2) = 12C_x, k_{11} (2,5) = 6C_x l,
\]

\[
k_{11} (3,3) = 12C_w + 6/5 C_y, k_{11} (3,6) = 6C_w l + 1/10 C_x l,
\]

\[
k_{11} (4,1) = 6C_y l, k_{11} (4,4) = 4C_y l^2, k_{11} (5,2) = 6C_x l, k_{11} (5,5) = 4C_x l^2,
\]

\[
k_{11} (6,3) = 6C_w l + 1/10 C_x l, k_{11} (6,6) = 4C_w l^2 + 2/15 C_x l^2,
\]

\[
k_{12} (1,1) = -12C_y, k_{12} (1,4) = 6C_y l, k_{12} (2,2) = -12C_x, k_{12} (2,5) = 6C_x l,
\]

\[
k_{12} (3,3) = -12C_w - 6/5 C_y, k_{12} (3,6) = 6C_w l + 1/10 C_x l,
\]

\[
k_{12} (4,1) = -6C_y l, k_{12} (4,4) = 2C_y l^2, k_{12} (5,2) = -6C_x l, k_{12} (5,5) = 2C_x l^2,
\]

\[
k_{12} (6,3) = -6C_w l - 1/10 C_x l, k_{12} (6,6) = 2C_w l^2 - 1/30 C_x l^2,
\]

\[
k_{21} (1,1) = -12C_y, k_{21} (1,4) = -6C_y l, k_{21} (2,2) = -12C_x, k_{21} (2,5) = -6C_x l,
\]

\[
k_{21} (3,3) = -12C_w - 6/5 C_y, k_{21} (3,6) = -6C_w l - 1/10 C_x l,
\]

\[
k_{21} (4,1) = 6C_y l, k_{21} (4,4) = 2C_y l^2, k_{21} (5,2) = 6C_x l, k_{21} (5,5) = 2C_x l^2,
\]

\[
k_{21} (6,3) = 6C_w l + 1/10 C_x l, k_{21} (6,6) = 2C_w l^2 - 1/30 C_x l^2,
\]
\[ k_{22} (1,1) = 12C_y, \quad k_{22} (1,4) = -6C_y l, \quad \]
\[ k_{22} (2,2) = 12C_x, \quad k_{22} (2,5) = -6C_x l, \]
\[ k_{22} (3,3) = 12C_w + 6/5 C_y, \quad k_{22} (3,6) = -6C_y l - 1/10 C_x l, \]
\[ k_{22} (4,1) = -6C_y l, \quad k_{22} (4,4) = 4C_y l^2, \quad k_{22} (5,2) = -6C_y l, \quad k_{22} (5,5) = 4C_y l^2, \]
\[ k_{22} (6,3) = -6C_w l - 1/10 C_x l, \quad k_{22} (6,6) = 4C_w l^2 + 2/15 C_x l^2, \]
\[ k_{13} (3, j + 2 p) = \frac{36 (1 + (-1)^j) C_i}{j^3 \pi^3}, \quad k_{13} (6, j + 2 p) = \frac{4 \left(5 + 4 (-1)^j\right) C_i l}{j^3 \pi^3}, \]
\[ k_{23} (3, j + 2 p) = \frac{-36 (1 + (-1)^j) C_i}{j^3 \pi^3}, \quad k_{23} (6, j + 2 p) = \frac{4 \left(4 + 5 (-1)^j\right) C_i l}{j^3 \pi^3}, \]
\[ k_{31} = (k_{13})^T, \quad k_{32} = (k_{23})^T, \]
\[ k_{33} (i, j) = \begin{cases} \frac{48 \left(1 + (-1)^j\right) i^3 j^3 \left(i^2 + j^2\right) C_y}{\left(i^2 - j^2\right)^4} & i \neq j, \\
\frac{3}{4} \frac{i^4 \pi^4}{60} + i^2 \pi^2 C_y & i = j, \end{cases} \]
\[ k_{33} (i + p, j + p) = \begin{cases} \frac{-48 \left(1 + (-1)^j\right) i^3 j^3 \left(i^2 + j^2\right) C_x}{\left(i^2 - j^2\right)^4} & i \neq j, \\
\frac{3}{4} \frac{i^4 \pi^4}{60} + i^2 \pi^2 C_x & i = j, \end{cases} \]
\[ k_{33} (i + 2 p, j + 2 p) = \begin{cases} \frac{16 \left(1 + (-1)^j\right) \left(i^4 + 4 i^2 j^2 + j^4\right) + 3 C_w i^2 j^2 \left(i^2 + j^2\right) \pi^2}{\left(i^2 - j^2\right)^4 \pi^2} & i \neq j, \\
\frac{1}{60} \left(C_i \left(10 - \frac{15}{i^2 \pi^2} + i^2 \pi^2\right) + C_w \left(45 + 60 i^2 \pi^2 + i^4 \pi^4\right)\right) & i = j, \end{cases} \]
and \[ C_y = EI_y, \quad C_x = EI_x, \quad C_y = GJ l^2, \quad C_w = EI_w, \quad i = 1, 2, \ldots, p, \quad j = 1, 2, \ldots, p. \]
The entries of the mass matrix are

\[
M^e = \rho I \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix},
\]

where

\[
m_{11}(1,1) = 13/35A, \quad m_{11}(1,4) = 11/210AI, \quad m_{11}(2,2) = 13/35A,
\]

\[
m_{11}(2,5) = 11/210AI, \quad m_{11}(3,3) = 13/35J, \quad m_{11}(3,6) = 11/210Jl,
\]

\[
m_{11}(4,1) = 11/210AI, \quad m_{11}(4,4) = 1/105Al^2, \quad m_{11}(5,2) = 11/210AI,
\]

\[
m_{11}(5,5) = 1/105Al^2, \quad m_{11}(6,3) = 11/210Jl, \quad m_{11}(6,6) = 1/105Jl^2,
\]

\[
m_{12}(1,1) = 9/70A, \quad m_{12}(1,4) = -13/420AI, \quad m_{12}(2,2) = 9/70A,
\]

\[
m_{12}(2,5) = -13/420AI, \quad m_{12}(3,3) = 9/70J, \quad m_{12}(3,6) = -13/420Jl,
\]

\[
m_{12}(4,1) = 13/420AI, \quad m_{12}(4,4) = -1/140Al^2, \quad m_{12}(5,2) = 13/420AI,
\]

\[
m_{12}(5,5) = -1/140Al^2, \quad m_{12}(6,3) = 13/420Jl, \quad m_{12}(6,6) = -1/140Jl^2,
\]

\[
m_{21}(1,1) = 9/70A, \quad m_{21}(1,4) = 13/420AI, \quad m_{21}(2,2) = 9/70A,
\]

\[
m_{21}(2,5) = 13/420AI, \quad m_{21}(3,3) = 9/70J, \quad m_{21}(3,6) = 13/420Jl,
\]

\[
m_{21}(4,1) = -13/420AI, \quad m_{21}(4,4) = -1/140Al^2, \quad m_{21}(5,2) = -13/420AI,
\]

\[
m_{21}(5,5) = -1/140Al^2, \quad m_{21}(6,3) = -13/420Jl, \quad m_{21}(6,6) = -1/140Jl^2,
\]

\[
m_{22}(1,1) = 13/35A, \quad m_{22}(1,4) = -11/210AI, \quad m_{22}(2,2) = 13/35A,
\]

\[
m_{22}(2,5) = -11/210AI, \quad m_{22}(3,3) = 13/35J, \quad m_{22}(3,6) = -11/210Jl,
\]

\[
m_{22}(4,1) = -11/210AI, \quad m_{22}(4,4) = 1/105Al^2, \quad m_{22}(5,2) = -11/210AI,
\]
\[ m_{22}(5, 5) = \frac{1}{105}AI^2, \quad m_{22}(6, 3) = -\frac{11}{210}Jl, \quad m_{22}(6, 6) = \frac{1}{105}Jl^2, \]

\[
m_{13}(1, j) = m_{13}(2, j + p) = \frac{2A\left(60(1 + (-1)^j) + j^2\pi^2\right)}{j^5\pi^5},
\]

\[
m_{13}(3, j + 2p) = \frac{2J\left(60(1 + (-1)^j) + j^2\pi^2\right)}{j^5\pi^5},
\]

\[
m_{13}(4, j) = m_{13}(5, j + p) = \frac{2AI\left(36 + 24(-1)^j - j^2\pi^2\right)}{j^5\pi^5},
\]

\[
m_{13}(6, j + 2p) = \frac{2JI\left(36 + 24(-1)^j - j^2\pi^2\right)}{j^5\pi^5},
\]

\[
m_{23}(1, j) = m_{23}(2, j + p) = -\frac{2A\left(60(1 + (-1)^j) + (-1)^j j^2\pi^2\right)}{j^5\pi^5},
\]

\[
m_{23}(3, j + 2p) = -\frac{2J\left(60(1 + (-1)^j) + (-1)^j j^2\pi^2\right)}{j^5\pi^5},
\]

\[
m_{23}(4, j) = m_{23}(5, j + p) = -\frac{2AI\left(-24 - 36(-1)^j + (-1)^j j^2\pi^2\right)}{j^5\pi^5},
\]

\[
m_{23}(6, j + 2p) = -\frac{2JI\left(-24 - 36(-1)^j + (-1)^j j^2\pi^2\right)}{j^5\pi^5},
\]

\[
m_{31} = (m_{13})^\top, \quad m_{32} = (m_{23})^\top,
\]

\[
m_{33}(i, j) = m_{33}(i + p, j + p) = \begin{cases} 
-\frac{48Aij\left(1 + (-1)^{i+j}\right)(i^2 + j^2)}{(i^2 - j^2)^4\pi^4} & \text{if } i \neq j, \\
\frac{1}{60}A\left(1 + \frac{45}{i^3\pi^4}\right) & \text{if } i = j,
\end{cases}
\]
\begin{equation}
\mathbf{m}_{33}(i+2p, j+2p) = 
\begin{cases} 
\frac{48Jij(1+(-1)^{i+j})(i^2+j^2)}{(i^2-j^2)^3 \pi^4} & i \neq j, \\
\frac{1}{60J}\left(1+\frac{45}{i^2 \pi^4}\right) & i = j,
\end{cases}
\end{equation}

and \( i = 1, 2, \ldots, p, \ j = 1, 2, \ldots, p. \)

The entries of the geometric matrix due to axial loads are

\begin{equation}
\mathbf{G}_p = \frac{1}{l}\begin{bmatrix}
g_{11}^p & g_{12}^p & g_{13}^p \\
g_{21}^p & g_{22}^p & g_{23}^p \\
g_{31}^p & g_{32}^p & g_{33}^p 
\end{bmatrix},
\end{equation}

where

\begin{align*}
g_{11}^p (1,1) &= 6/5, \ g_{11}^p (1,4) = 1/10l, \ g_{11}^p (2,2) = 6/5, \ g_{11}^p (2,5) = 1/10l, \\
g_{11}^p (4,1) &= 1/10l, \ g_{11}^p (4,4) = 2/15l^2, \ g_{11}^p (5,2) = 1/10l, \ g_{11}^p (5,5) = 2/15l^2, \\
g_{12}^p (1,1) &= -6/5, \ g_{12}^p (1,4) = 1/10l, \ g_{12}^p (2,2) = -6/5, \ g_{12}^p (2,5) = 1/10l, \\
g_{12}^p (4,1) &= -1/10l, \ g_{12}^p (4,4) = -1/30l^2, \ g_{12}^p (5,2) = -1/10l, \ g_{12}^p (5,5) = -1/30l^2, \\
g_{13}^p (1,1) &= -6/5, \ g_{13}^p (1,4) = -1/10l, \ g_{13}^p (2,2) = -6/5, \ g_{13}^p (2,5) = -1/10l, \\
g_{13}^p (4,1) &= 1/10l, \ g_{13}^p (4,4) = -1/30l^2, \ g_{13}^p (5,2) = 1/10l, \ g_{13}^p (5,5) = -1/30l^2, \\
g_{21}^p (1,1) &= 6/5, \ g_{21}^p (1,4) = -1/10l, \ g_{21}^p (2,2) = 6/5, \ g_{21}^p (2,5) = -1/10l, \\
g_{21}^p (4,1) &= 1/10l, \ g_{21}^p (4,4) = -1/30l^2, \ g_{21}^p (5,2) = 1/10l, \ g_{21}^p (5,5) = -1/30l^2, \\
g_{22}^p (1,1) &= 6/5, \ g_{22}^p (1,4) = -1/10l, \ g_{22}^p (2,2) = 6/5, \ g_{22}^p (2,5) = -1/10l, \\
g_{22}^p (4,1) &= -1/10l, \ g_{22}^p (4,4) = 2/15l^2, \ g_{22}^p (5,2) = -1/10l, \ g_{22}^p (5,5) = 2/15l^2, \\
g_{13}^p (1, j) &= g_{13}^p (2, j + p) = \frac{36\left(1+(-1)^j\right)}{j^2 \pi^3},
\end{align*}
\[
\begin{align*}
g^{p}_{13}(4, j) &= g^{p}_{13}(5, j + p) = \frac{4l \left(5 + 4(-1)^{j} \right)}{j^3 \pi^3}, \\
g^{p}_{23}(1, j) &= g^{p}_{23}(2, j + p) = -\frac{36 \left(1 + (-1)^{j} \right)}{j^3 \pi^3}, \\
g^{p}_{23}(4, j) &= g^{p}_{23}(5, j + p) = \frac{4l \left(4 + 5(-1)^{j} \right)}{j^3 \pi^3}, \\
g^{p}_{31} &= (g^{p}_{13})^{T}, g^{p}_{12} = (g^{p}_{23})^{T}, \\
g^{p}_{33}(i, j) &= g^{p}_{33}(i + p, j + p) = \\
&\begin{cases}
-\frac{16ij \left(1 + (-1)^{i+j} \right)(i^4 + 4i^2 j^2 + j^4)}{(i^2 - j^2)^4 \pi^4} & i \neq j, \\
\frac{1}{60} \left(10 - \frac{15}{i^2 \pi^2} + \frac{j^2 \pi^2}{i^2 \pi^2} \right) & i = j,
\end{cases}
\end{align*}
\]

and \( i = 1, 2, \ldots, p, \quad j = 1, 2, \ldots, p \).

The entries of the geometric matrix due to applied torque are

\[
G^{e}_{L} = \frac{1}{Jl^{2}} \begin{bmatrix}
g^{L}_{11} & g^{L}_{12} & g^{L}_{13} \\
g^{L}_{21} & g^{L}_{22} & g^{L}_{23} \\
g^{L}_{31} & g^{L}_{32} & g^{L}_{33}
\end{bmatrix}, \quad (A3.4)
\]

where

\[
\begin{align*}
g^{L}_{11}(1, 5) &= -(I_x + I_y)l, \quad g^{L}_{11}(2, 4) = (I_x + I_y)l, \quad g^{L}_{11}(4, 2) = (I_x + I_y)l, \\
g^{L}_{11}(4, 5) &= \frac{1}{2} \left( -I_x - I_y \right)l^2, \quad g^{L}_{11}(5, 1) = -(I_x + I_y)l, \quad g^{L}_{11}(5, 4) = \frac{1}{2} \left( I_x - I_y \right)l^2, \\
g^{L}_{12}(1, 5) &= (I_x + I_y)l, \quad g^{L}_{12}(2, 4) = -(I_x + I_y)l, \quad g^{L}_{12}(4, 2) = -(I_x + I_y)l, \\
g^{L}_{12}(4, 5) &= \frac{1}{2} \left( I_x + I_y \right)l^2, \quad g^{L}_{12}(5, 1) = (I_x + I_y)l, \quad g^{L}_{12}(5, 4) = -\frac{1}{2} \left( I_x + I_y \right)l^2, \\
g^{L}_{21}(1, 5) &= (I_x + I_y)l, \quad g^{L}_{21}(2, 4) = -(I_x + I_y)l, \quad g^{L}_{21}(4, 2) = -(I_x + I_y)l, \\
g^{L}_{22}(4, 5) &= \frac{1}{2} \left( I_x + I_y \right)l^2, \quad g^{L}_{22}(5, 1) = (I_x + I_y)l, \quad g^{L}_{22}(5, 4) = -\frac{1}{2} \left( I_x + I_y \right)l^2,
\end{align*}
\]
\[
\begin{align*}
\mathbf{g}_{21}^L(4,5) &= -\frac{1}{2} \left( I_x + I_y \right) L^2, \quad \mathbf{g}_{21}^L(5,1) = \left( I_x + I_y \right) I, \quad \mathbf{g}_{21}^L(5,4) = -\frac{1}{2} \left( I_x + I_y \right) I^2, \\
\mathbf{g}_{22}^L(1,5) &= -\left( I_x + I_y \right) I, \quad \mathbf{g}_{22}^L(2,4) = \left( I_x + I_y \right) I, \quad \mathbf{g}_{22}^L(4,2) = \left( I_x + I_y \right) I, \\
\mathbf{g}_{22}^L(4,5) &= -\frac{1}{2} \left( I_x - I_y \right) I^2, \quad \mathbf{g}_{22}^L(5,1) = -\left( I_x + I_y \right) I, \quad \mathbf{g}_{22}^L(5,4) = -\frac{1}{2} \left( I_x - I_y \right) I^2, \\
\mathbf{g}_{31}^L(1, j + p) &= -\mathbf{g}_{33}^L(2, j) = -\mathbf{g}_{31}^L(1, j + p) = \mathbf{g}_{33}^L(2, j) = \frac{24 \left( -1 + \left( \frac{-1}{j^3 \pi^3} \right) \left( I_x + I_y \right) \right)}{j^3 \pi^3}, \\
\mathbf{g}_{31}^L(4, j + p) &= -\mathbf{g}_{33}^L(5, j) = \mathbf{g}_{33}^L(4, j + p) = -\mathbf{g}_{33}^L(5, j) = \frac{12I \left( -1 + \left( \frac{-1}{j^3 \pi^3} \right) \left( I_x + I_y \right) \right)}{j^3 \pi^3}, \\
\mathbf{g}_{31}^L &= \left( \mathbf{g}_{33}^L \right)^T, \quad \mathbf{g}_{32}^L = \left( \mathbf{g}_{31}^L \right)^T, \\
\mathbf{g}_{31}^L(i, j + p) &= -\mathbf{g}_{33}^L(i + p, j) \\
&= \begin{cases} 
4 \left( -1 + \left( \frac{-1}{j^3 \pi^3} \right) \left( I_x + I_y \right) \right) \left( -3j^6 + i^2 j^4 \left( -45 + 2j^2 \pi^2 \right) \\
+ i^4 \left( 3 + 2j^2 \pi^2 \right) - i^4 j^2 \left( 45 + 4j^2 \pi^2 \right) \right) \\
\left( i^2 - j^2 \right)^3 \pi^5 \\
0 
\end{cases} 
\quad \text{for } i \neq j,
\end{align*}
\]

and \( i = 1, 2, \ldots, p, \quad j = 1, 2, \ldots, p \).

The entries of the geometric matrix due to pure bending moment \( M \) are

\[
\mathbf{G}_M^* = \frac{1}{I} \begin{bmatrix}
\mathbf{g}_{11}^M & \mathbf{g}_{12}^M & \mathbf{g}_{13}^M \\
\mathbf{g}_{21}^M & \mathbf{g}_{22}^M & \mathbf{g}_{23}^M \\
\mathbf{g}_{31}^M & \mathbf{g}_{32}^M & \mathbf{g}_{33}^M
\end{bmatrix},
\]

where

\[
\begin{align*}
\mathbf{g}_{11}^M(1,3) &= 6/5, \quad \mathbf{g}_{11}^M(1,6) = 1/10I, \quad \mathbf{g}_{11}^M(3,1) = 6/5, \quad \mathbf{g}_{11}^M(3,4) = 1/10I, \\
\mathbf{g}_{11}^M(4,3) &= 1/10I, \quad \mathbf{g}_{11}^M(4,6) = 2/15I^2, \quad \mathbf{g}_{11}^M(6,1) = 1/10I, \quad \mathbf{g}_{11}^M(6,4) = 2/15I^2,
\end{align*}
\]
\[ \mathbf{g}^M_{11} (1, 3) = -6/5, \mathbf{g}^M_{13} (1, 6) = 1/10 \ell, \mathbf{g}^M_{12} (3, 1) = -6/5, \mathbf{g}^M_{12} (3, 4) = 1/10 \ell, \]
\[ \mathbf{g}^M_{12} (4, 3) = -1/10 \ell, \mathbf{g}^M_{16} (4, 6) = -1/30 \ell^2, \mathbf{g}^M_{12} (6, 1) = -1/10 \ell, \mathbf{g}^M_{12} (6, 4) = -1/30 \ell^2, \]
\[ \mathbf{g}^M_{21} (1, 3) = -6/5, \mathbf{g}^M_{21} (1, 6) = -1/10 \ell, \mathbf{g}^M_{21} (3, 1) = -6/5, \mathbf{g}^M_{21} (3, 4) = -1/10 \ell, \]
\[ \mathbf{g}^M_{21} (4, 3) = 1/10 \ell, \mathbf{g}^M_{21} (4, 6) = -1/30 \ell^2, \mathbf{g}^M_{21} (6, 1) = 1/10 \ell, \mathbf{g}^M_{21} (6, 4) = -1/30 \ell^2, \]
\[ \mathbf{g}^M_{22} (1, 3) = 6/5, \mathbf{g}^M_{22} (1, 6) = -1/10 \ell, \mathbf{g}^M_{22} (3, 1) = 6/5, \mathbf{g}^M_{22} (3, 4) = -1/10 \ell, \]
\[ \mathbf{g}^M_{22} (4, 3) = -1/10 \ell, \mathbf{g}^M_{22} (4, 6) = 2/15 \ell^2, \mathbf{g}^M_{22} (6, 1) = -1/10 \ell, \mathbf{g}^M_{22} (6, 4) = 2/15 \ell^2, \]
\[ \mathbf{g}^M_{13} (1, j + 2p) = \mathbf{g}^M_{13} (3, j) = \frac{36(1 + (-1)^j)}{j^3 \pi^3}, \]
\[ \mathbf{g}^M_{13} (4, j + 2p) = \mathbf{g}^M_{13} (6, j) = -\frac{4l(5 + 4(-1)^j)}{j^3 \pi^3}, \]
\[ \mathbf{g}^M_{23} (1, j + 2p) = \mathbf{g}^M_{23} (3, j) = -\frac{36(1 + (-1)^j)}{j^3 \pi^3}, \]
\[ \mathbf{g}^M_{23} (4, j + 2p) = \mathbf{g}^M_{23} (6, j) = -\frac{4l(4 + 5(-1)^j)}{j^3 \pi^3}, \]
\[ \mathbf{g}^M = (\mathbf{g}^M_{13})^T, \mathbf{g}^M = (\mathbf{g}^M_{23})^T, \]
\[ \mathbf{g}^M_{33} (i, j + 2p) = \mathbf{g}^M_{33} (i + 2p, j) = \begin{cases} 
16ij(1 + (-1)^j)(i^4 + 4i^2j^2 + j^4) & i \neq j, \\
\frac{1}{60}
\begin{pmatrix} 
15 & 1 \end{pmatrix} & i = j,
\end{cases}
\]
and \( i = 1, 2, \cdots, p, \ j = 1, 2, \cdots, p. \)
The entries of the geometric matrix due to pure bending moment $N$ are

$$G_N^e = \frac{1}{l} \begin{bmatrix} g_{11}^N & g_{12}^N & g_{13}^N \\ g_{21}^N & g_{22}^N & g_{23}^N \\ g_{31}^N & g_{32}^N & g_{33}^N \end{bmatrix},$$

(A3.6)

where

$$g_{11}^N (2,3) = 6/5, g_{11}^N (2,6) = 1/10l, g_{11}^N (3,2) = 6/5, g_{11}^N (3,5) = 1/10l,$$

$$g_{11}^N (5,3) = 1/10l, g_{11}^N (5,6) = 2/15l^2, g_{11}^N (6,2) = 1/10l, g_{11}^N (6,5) = 2/15l^2,$$

$$g_{12}^N (2,3) = -6/5, g_{12}^N (2,6) = 1/10l, g_{12}^N (3,2) = -6/5, g_{12}^N (3,5) = 1/10l,$$

$$g_{12}^N (5,3) = -1/10l, g_{12}^N (5,6) = -1/30l^2, g_{12}^N (6,2) = -1/10l, g_{12}^N (6,5) = -1/30l^2,$$

$$g_{21}^N (2,3) = -6/5, g_{21}^N (2,6) = -1/10l, g_{21}^N (3,2) = -6/5, g_{21}^N (3,5) = -1/10l,$$

$$g_{21}^N (5,3) = 1/10l, g_{21}^N (5,6) = -1/30l^2, g_{21}^N (6,2) = 1/10l, g_{21}^N (6,5) = -1/30l^2,$$

$$g_{22}^N (2,3) = 6/5, g_{22}^N (2,6) = -1/10l, g_{22}^N (3,2) = 6/5, g_{22}^N (3,5) = -1/10l,$$

$$g_{22}^N (5,3) = -1/10l, g_{22}^N (5,6) = 2/15l^2, g_{22}^N (6,2) = -1/10l, g_{22}^N (6,5) = 2/15l^2,$$

$$g_{13}^N (2, j + 2p) = g_{13}^N (3, j + p) = \frac{36(1 + (-1)^j)}{j^3\pi^3},$$

$$g_{15}^N (5, j + 2p) = g_{15}^N (6, j + p) = \frac{4l(5 + 4(-1)^j)}{j^3\pi^3},$$

$$g_{23}^N (2, j + 2p) = g_{23}^N (3, j + p) = \frac{-36(1 + (-1)^j)}{j^3\pi^3},$$

$$g_{25}^N (5, j + 2p) = g_{25}^N (6, j + p) = \frac{4l(4 + 5(-1)^j)}{j^3\pi^3},$$
\[ g_1^N = \left( g_{13}^N \right)^T, \ g_{12}^N = \left( g_{23}^N \right)^T, \]

\[ g_{33}^N (i + p, j + 2p) = g_{33}^N (i + 2p, j + p) = \begin{cases} 
16ij \left( 1 + (-1)^{i+j} \right) \left( i^4 + 4i^2 j^2 + j^4 \right) & i \neq j, \\
\frac{1}{60} \left( 10 - \frac{15}{i^2 \pi^2} + i^2 \pi^2 \right) & i = j,
\end{cases} \]

and \( i = 1, 2, \ldots, p, \ j = 1, 2, \ldots, p. \)
APPENDIX 4: MATRIX ENTRIES FOR PRE-TWISTED STRAIGHT BEAMS

The non-zero coefficients of the stiffness matrix $\mathbf{K}^e$ are:

<table>
<thead>
<tr>
<th>$m$</th>
<th>$n$</th>
<th>$\mathbf{K}^e_{m,n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$6i-5$</td>
<td>$6j-5$</td>
<td>$A_{i,j}^{0,0} \cdot \bar{\mu}^2 \cdot g + A_{i,j}^{1,1} \cdot g$</td>
</tr>
<tr>
<td>$6i-4$</td>
<td>$6j-5$</td>
<td>$-A_{i,j}^{0,0} \cdot \bar{\mu} \cdot g + A_{i,j}^{0,1} \cdot \bar{\mu} \cdot g$</td>
</tr>
<tr>
<td>$6i-2$</td>
<td>$6j-5$</td>
<td>$-A_{i,j}^{0,0} \cdot \bar{\mu} \cdot g$</td>
</tr>
<tr>
<td>$6i-1$</td>
<td>$6j-5$</td>
<td>$-A_{i,j}^{0,1} \cdot g$</td>
</tr>
<tr>
<td>$6i-5$</td>
<td>$6j-4$</td>
<td>$A_{i,j}^{1,0} \cdot \bar{\mu} \cdot g - A_{i,j}^{0,1} \cdot \bar{\mu} \cdot g$</td>
</tr>
<tr>
<td>$6i-4$</td>
<td>$6j-4$</td>
<td>$A_{i,j}^{0,0} \cdot \bar{\mu}^2 \cdot g + A_{i,j}^{1,1} \cdot g$</td>
</tr>
<tr>
<td>$6i-2$</td>
<td>$6j-4$</td>
<td>$A_{i,j}^{1,1} \cdot g$</td>
</tr>
<tr>
<td>$6i-1$</td>
<td>$6j-4$</td>
<td>$-A_{i,j}^{0,0} \cdot \bar{\mu} \cdot g$</td>
</tr>
<tr>
<td>$6i-3$</td>
<td>$6j-3$</td>
<td>$A_{i,j}^{1,1} \cdot g$</td>
</tr>
</tbody>
</table>

The non-zero coefficients of the mass matrix $\mathbf{M}^e$ are:

<table>
<thead>
<tr>
<th>$m$</th>
<th>$n$</th>
<th>$\mathbf{M}^e_{m,n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$6i-5$</td>
<td>$6j-5$</td>
<td>$A_{i,j}^{0,0}$</td>
</tr>
<tr>
<td>$6i-4$</td>
<td>$6j-4$</td>
<td>$A_{i,j}^{0,0}$</td>
</tr>
<tr>
<td>$6i-3$</td>
<td>$6j-3$</td>
<td>$A_{i,j}^{0,0}$</td>
</tr>
<tr>
<td>$6i-2$</td>
<td>$6j-2$</td>
<td>$A_{i,j}^{0,0} \cdot r_1$</td>
</tr>
<tr>
<td>$6i-1$</td>
<td>$6j-1$</td>
<td>$A_{i,j}^{0,0} \cdot r_2$</td>
</tr>
<tr>
<td>$6i$</td>
<td>$6j$</td>
<td>$A_{i,j}^{0,0} \cdot r_0$</td>
</tr>
</tbody>
</table>
The non-zero coefficients of the geometric stiffness matrix $G_{q_i}$ are:

<table>
<thead>
<tr>
<th>$m$</th>
<th>$n$</th>
<th>$G_{m,n}^e$</th>
<th>$m$</th>
<th>$n$</th>
<th>$G_{m,n}^e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$6i - 5$</td>
<td>$6j - 5$</td>
<td>$A_{i,j}^{0,0} \cdot \bar{\mu}^2 + A_{i,j}^{1,1}$</td>
<td>$6i - 2$</td>
<td>$6j - 2$</td>
<td>$A_{i,j}^{0,0} \cdot r_2 \cdot \bar{\mu}^2 + A_{i,j}^{1,1} \cdot r_1$</td>
</tr>
<tr>
<td>$6i - 4$</td>
<td>$6j - 5$</td>
<td>$-A_{i,j}^{1,0} \cdot \bar{\mu} + A_{i,j}^{0,1} \cdot \bar{\mu}$</td>
<td>$6i - 1$</td>
<td>$6j - 2$</td>
<td>$-A_{i,j}^{1,0} \cdot \bar{\mu} \cdot r_2 + A_{i,j}^{0,1} \cdot \bar{\mu} \cdot r_1$</td>
</tr>
<tr>
<td>$6i - 5$</td>
<td>$6j - 4$</td>
<td>$A_{i,j}^{1,0} \cdot \bar{\mu} - A_{i,j}^{0,1} \cdot \bar{\mu}$</td>
<td>$6i - 2$</td>
<td>$6j - 1$</td>
<td>$A_{i,j}^{1,0} \cdot \bar{\mu} \cdot r_2 - A_{i,j}^{0,1} \cdot \bar{\mu} \cdot r_1$</td>
</tr>
<tr>
<td>$6i - 4$</td>
<td>$6j - 4$</td>
<td>$A_{i,j}^{0,0} \cdot \bar{\mu}^2 + A_{i,j}^{1,1}$</td>
<td>$6i - 1$</td>
<td>$6j - 1$</td>
<td>$A_{i,j}^{0,0} \cdot r_1 \cdot \bar{\mu}^2 + A_{i,j}^{1,1} \cdot r_2$</td>
</tr>
<tr>
<td>$6i - 3$</td>
<td>$6j - 3$</td>
<td>$A_{i,j}^{1,1}$</td>
<td>$6i$</td>
<td>$6j$</td>
<td>$A_{i,j}^{1,1}\cdot r_2$</td>
</tr>
</tbody>
</table>

The non-zero coefficients of the geometric stiffness matrix $G_{r_i}$ are:

<table>
<thead>
<tr>
<th>$m$</th>
<th>$n$</th>
<th>$G_{m,n}^e$</th>
<th>$m$</th>
<th>$n$</th>
<th>$G_{m,n}^e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$6i - 2$</td>
<td>$6j - 2$</td>
<td>$2\cdot A_{i,j}^{0,0} \cdot \bar{\mu} \cdot \bar{\mu}^2 / r_i$</td>
<td>$6i - 2$</td>
<td>$6j - 1$</td>
<td>$A_{i,j}^{1,0} - A_{i,j}^{1,1} \cdot r_2 / r_1$</td>
</tr>
<tr>
<td>$6i - 1$</td>
<td>$6j - 2$</td>
<td>$-A_{i,j}^{1,0} \cdot \bar{\mu} \cdot \bar{\mu}^2 / r_i + A_{i,j}^{0,1}$</td>
<td>$6i - 1$</td>
<td>$6j - 1$</td>
<td>$2\cdot A_{i,j}^{0,0} \cdot \bar{\mu}$</td>
</tr>
</tbody>
</table>

The non-zero coefficients of the geometric stiffness matrix $G_{q_i}$ are:

<table>
<thead>
<tr>
<th>$m$</th>
<th>$n$</th>
<th>$G_{m,n}^e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$6i$</td>
<td>$6j - 5$</td>
<td>$-B_{i,j}^{0,0} \cdot \bar{\mu} + C_{i,j}^{1,0} \cdot \bar{\mu} + D_{i,j}^{0,1} - E_{i,j}^{1,1}$</td>
</tr>
<tr>
<td>$6i$</td>
<td>$6j - 4$</td>
<td>$D_{i,j}^{0,0} \cdot \bar{\mu} - E_{i,j}^{1,0} \cdot \bar{\mu} + B_{i,j}^{0,1} - C_{i,j}^{1,1}$</td>
</tr>
<tr>
<td>$6i - 2$</td>
<td>$6j - 3$</td>
<td>$-D_{i,j}^{0,1} - C_{i,j}^{0,1} \cdot \bar{\mu} + E_{i,j}^{1,1}$</td>
</tr>
<tr>
<td>$6i - 1$</td>
<td>$6j - 3$</td>
<td>$E_{i,j}^{0,1} \cdot \bar{\mu} - B_{i,j}^{0,1} + C_{i,j}^{1,1}$</td>
</tr>
<tr>
<td>$6i - 3$</td>
<td>$6j - 2$</td>
<td>$-D_{i,j}^{0,1} - C_{i,j}^{1,0} \cdot \bar{\mu} + E_{i,j}^{1,1}$</td>
</tr>
<tr>
<td>$6i - 3$</td>
<td>$6j - 1$</td>
<td>$E_{i,j}^{0,1} \cdot \bar{\mu} - B_{i,j}^{1,0} + C_{i,j}^{1,1}$</td>
</tr>
<tr>
<td>$6i - 5$</td>
<td>$6j$</td>
<td>$-B_{i,j}^{0,0} \cdot \bar{\mu} + D_{i,j}^{0,1} + C_{i,j}^{0,1} \cdot \bar{\mu} - E_{i,j}^{1,1}$</td>
</tr>
<tr>
<td>$6i - 4$</td>
<td>$6j$</td>
<td>$D_{i,j}^{0,0} \cdot \bar{\mu} + B_{i,j}^{1,0} - E_{i,j}^{0,1} \cdot \bar{\mu} - C_{i,j}^{1,1}$</td>
</tr>
</tbody>
</table>
The non-zero coefficients of the geometric stiffness matrix $\bar{G}_{e_{ij}}$ are:

<table>
<thead>
<tr>
<th>$m$</th>
<th>$n$</th>
<th>$\bar{G}_{m,n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6i</td>
<td>6j-5</td>
<td>$-D_{i,j}^{0,0} \cdot \bar{\mu} + E_{i,j}^{1,0} \cdot \bar{\mu} - B_{i,j}^{1,1}$</td>
</tr>
<tr>
<td>6i</td>
<td>6j-4</td>
<td>$-B_{i,j}^{0,0} \cdot \bar{\mu} + C_{i,j}^{1,0} \cdot \bar{\mu} + D_{i,j}^{1,0} - E_{i,j}^{1,1}$</td>
</tr>
<tr>
<td>6i-2</td>
<td>6j-3</td>
<td>$B_{i,j}^{0,0} - E_{i,j}^{0,1} \cdot \bar{\mu} - C_{i,j}^{1,1}$</td>
</tr>
<tr>
<td>6i-1</td>
<td>6j-3</td>
<td>$-C_{i,j}^{0,1} \cdot \bar{\mu} - D_{i,j}^{0,1} + E_{i,j}^{1,1}$</td>
</tr>
<tr>
<td>6i-3</td>
<td>6j-2</td>
<td>$B_{i,j}^{0,0} - E_{i,j}^{0,1} \cdot \bar{\mu} - C_{i,j}^{1,1}$</td>
</tr>
<tr>
<td>6i-3</td>
<td>6j-1</td>
<td>$-C_{i,j}^{0,1} \cdot \bar{\mu} - D_{i,j}^{0,1} + E_{i,j}^{1,1}$</td>
</tr>
<tr>
<td>6i-5</td>
<td>6j</td>
<td>$-D_{i,j}^{0,0} \cdot \bar{\mu} - B_{i,j}^{1,0} + E_{i,j}^{0,1} \cdot \bar{\mu} + C_{i,j}^{1,1}$</td>
</tr>
<tr>
<td>6i-4</td>
<td>6j</td>
<td>$-B_{i,j}^{0,0} \cdot \bar{\mu} + D_{i,j}^{1,0} + C_{i,j}^{0,1} \cdot \bar{\mu} - E_{i,j}^{1,1}$</td>
</tr>
</tbody>
</table>

The non-zero coefficients of the geometric stiffness matrix $\bar{G}_{m_{ij}}$ are:

<table>
<thead>
<tr>
<th>$m$</th>
<th>$n$</th>
<th>$\bar{G}_{m,n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6i</td>
<td>6j-5</td>
<td>$-D_{i,j}^{0,0} \cdot \bar{\mu} - B_{i,j}^{1,1}$</td>
</tr>
<tr>
<td>6i</td>
<td>6j-4</td>
<td>$-B_{i,j}^{0,0} \cdot \bar{\mu} + D_{i,j}^{1,1}$</td>
</tr>
<tr>
<td>6i-2</td>
<td>6j-3</td>
<td>$D_{i,j}^{0,0} \cdot \bar{\mu} + B_{i,j}^{1,1}$</td>
</tr>
<tr>
<td>6i-1</td>
<td>6j-3</td>
<td>$B_{i,j}^{0,0} \cdot \bar{\mu} - D_{i,j}^{1,1}$</td>
</tr>
<tr>
<td>6i-3</td>
<td>6j-2</td>
<td>$D_{i,j}^{1,0} \cdot \bar{\mu} + B_{i,j}^{1,1}$</td>
</tr>
<tr>
<td>6i-3</td>
<td>6j-1</td>
<td>$B_{i,j}^{1,0} \cdot \bar{\mu} - D_{i,j}^{1,1}$</td>
</tr>
<tr>
<td>6i-5</td>
<td>6j</td>
<td>$-D_{i,j}^{0,1} \cdot \bar{\mu} - B_{i,j}^{1,1}$</td>
</tr>
<tr>
<td>6i-4</td>
<td>6j</td>
<td>$-B_{i,j}^{0,1} \cdot \bar{\mu} + D_{i,j}^{1,1}$</td>
</tr>
</tbody>
</table>

The non-zero coefficients of the geometric stiffness matrix $\bar{G}_{e_{ij}}$ are:

<table>
<thead>
<tr>
<th>$m$</th>
<th>$n$</th>
<th>$\bar{G}_{m,n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6i</td>
<td>6j-5</td>
<td>$-B_{i,j}^{1,0} \cdot \bar{\mu} - D_{i,j}^{1,1}$</td>
</tr>
<tr>
<td>6i</td>
<td>6j-4</td>
<td>$-D_{i,j}^{0,0} \cdot \bar{\mu} - B_{i,j}^{1,1}$</td>
</tr>
<tr>
<td>6i-2</td>
<td>6j-3</td>
<td>$-B_{i,j}^{1,0} \cdot \bar{\mu} + D_{i,j}^{1,1}$</td>
</tr>
<tr>
<td>6i-1</td>
<td>6j-3</td>
<td>$D_{i,j}^{0,0} \cdot \bar{\mu} + B_{i,j}^{1,1}$</td>
</tr>
<tr>
<td>6i-3</td>
<td>6j-2</td>
<td>$-B_{i,j}^{0,1} \cdot \bar{\mu} + D_{i,j}^{1,1}$</td>
</tr>
<tr>
<td>6i-3</td>
<td>6j-1</td>
<td>$B_{i,j}^{1,0} \cdot \bar{\mu} - D_{i,j}^{1,1}$</td>
</tr>
<tr>
<td>6i-5</td>
<td>6j</td>
<td>$B_{i,j}^{0,1} \cdot \bar{\mu} - D_{i,j}^{1,1}$</td>
</tr>
<tr>
<td>6i-4</td>
<td>6j</td>
<td>$-D_{i,j}^{0,1} \cdot \bar{\mu} - B_{i,j}^{1,1}$</td>
</tr>
</tbody>
</table>
The integrations are

\[ A_{i,j}^{\alpha,\beta} = \int_0^1 \mathbf{F}_i^{\alpha} \cdot \mathbf{F}_j^{\beta} d\xi, \]

\[ B_{i,j}^{\alpha,\beta} = \int_0^1 \cos[\bar{\mu} \cdot (1 - \xi)] \cdot \mathbf{F}_i^{\alpha} \cdot \mathbf{F}_j^{\beta} d\xi, \]

\[ C_{i,j}^{\alpha,\beta} = \int_0^1 \cos[\bar{\mu} \cdot (1 - \xi) \cdot (1 - \xi)] \cdot \mathbf{F}_i^{\alpha} \cdot \mathbf{F}_j^{\beta} d\xi, \]

\[ D_{i,j}^{\alpha,\beta} = \int_0^1 \sin[\bar{\mu} \cdot (1 - \xi)] \cdot \mathbf{F}_i^{\alpha} \cdot \mathbf{F}_j^{\beta} d\xi, \]

\[ E_{i,j}^{\alpha,\beta} = \int_0^1 \sin[\bar{\mu} \cdot (1 - \xi) \cdot (1 - \xi)] \cdot \mathbf{F}_i^{\alpha} \cdot \mathbf{F}_j^{\beta} d\xi, \]

where \( i = 1, 2, \cdots p + 2, \ j = 1, 2, \cdots p + 2, \) and the superscript \( \alpha \) and \( \beta \) (\( \alpha, \beta = 0, 1 \)) denote the order of the derivatives with respect to \( \xi, \) and \( \bar{\mu}, r_1, r_2, r_0, r_j, g \) are the non-dimensional parameters given in Eq. (4.30).
APPENDIX 5: MATRIX ENTRIES FOR TRAPIZOIDAL MINDLIN PLATES

Defined \( m = 3(j + (i-1)(q+2)) - 2 \) and \( n = 3(l + (k-1)(q+2)) - 2 \).

The non-zero coefficients of the geometric stiffness matrix

a) for plates under uniaxial compression on \( x \) direction are,

\[
G_{m,n}^e = c_0 A_{(1)i,k}^{1,1} B_{(1)j,l}^{0,0},
\]

b) for plates under isotropic in-plane compression are,

\[
G_{m,n}^e = c_0 A_{(1)i,k}^{1,1} B_{(1)j,l}^{0,0} + \frac{1}{c_0} A_{(3)i,k}^{1,1} B_{(3)j,l}^{0,0} - \frac{1}{c_0} A_{(2)i,k}^{1,0} B_{(2)j,l}^{1,1} - \frac{1}{c_0} A_{(2)i,k}^{1,0} B_{(2)j,l}^{1,1} + \frac{1}{c_0} A_{(2)i,k}^{0,1} B_{(2)j,l}^{1,1} + \frac{1}{c_0} A_{(1)i,k}^{0,0} B_{(1)j,l}^{1,1}.
\]

The integrations are

\[
A_{(\zeta,i,k)\alpha\beta} = \int_0^1 (b_0 + e_0 \xi)^{(\zeta-1)} f_i^{\alpha} f_k^{\beta} d\xi,
\]

\[
B_{(\zeta,i,k)\alpha\beta} = \int_0^1 (a_0 + e_0 \xi)^{(\zeta-2)} f_i^{\alpha} f_k^{\beta} d\xi,
\]

where \( i, k = 1, 2, \ldots, p + 2 \), \( j, l = 1, 2, \ldots, p + 2 \), \( \zeta = 1, 2, 3 \) and the superscripts \( \alpha \) and \( \beta \) (\( \alpha, \beta = 0, 1 \)) denote the order of the derivatives.
APPENDIX 6: MATRIX ENTRIES FOR CYLINDRICAL SHELL PANELS

Defined \( ji = j + (i-1)(p+2), \ lk = l + (k-1)(p+2). \)

The non-zero coefficients of the stiffness matrix \( K^e_{cy} \) are

<table>
<thead>
<tr>
<th>( m )</th>
<th>( n )</th>
<th>( K^e_{cy} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5ji - 4</td>
<td>5lk - 4</td>
<td>([L_x \cdot L_f \cdot A_{ij,k}^{1,0} \cdot B_{ij,j}^{0,0} \cdot C + L_x \cdot A_{ij,k}^{0,0} \cdot B_{ij,j}^{1,1} \cdot C_m]/a^2)</td>
</tr>
<tr>
<td>5ji - 4</td>
<td>5lk - 3</td>
<td>( A_{ij,k}^{1,0} \cdot B_{ij,j}^{0,1} \cdot \nu \cdot C/a + A_{ij,k}^{0,1} \cdot B_{ij,j}^{1,0} \cdot C_m/a )</td>
</tr>
<tr>
<td>5ji - 4</td>
<td>5lk - 2</td>
<td>(-L_f \cdot A_{ij,k}^{1,0} \cdot B_{ij,j}^{0,0} \cdot \nu \cdot C/a)</td>
</tr>
<tr>
<td>5ji - 3</td>
<td>5lk - 4</td>
<td>( A_{ij,k}^{1,0} \cdot B_{ij,j}^{0,1} \cdot \nu \cdot C/a + A_{ij,k}^{0,1} \cdot B_{ij,j}^{1,0} \cdot C_m/a )</td>
</tr>
<tr>
<td>5ji - 3</td>
<td>5lk - 3</td>
<td>([L_x \cdot L_f \cdot A_{ij,k}^{0,0} \cdot B_{ij,j}^{1,1} \cdot C/a^2 + L_x \cdot A_{ij,k}^{1,0} \cdot B_{ij,j}^{0,0} \cdot C_m + L_x \cdot L_f \cdot A_{ij,k}^{0,0} \cdot B_{ij,j}^{0,0} \cdot G_m/a^2 + L_f \cdot L_x \cdot A_{ij,k}^{1,0} \cdot B_{ij,j}^{0,0} \cdot D_m/a^2)</td>
</tr>
<tr>
<td>5ji - 3</td>
<td>5lk - 2</td>
<td>(-L_x \cdot A_{ij,k}^{0,0} \cdot B_{ij,j}^{1,0} \cdot C/a^2 + L_x \cdot A_{ij,k}^{0,0} \cdot B_{ij,j}^{0,1} \cdot G_m/a^2)</td>
</tr>
<tr>
<td>5ji - 3</td>
<td>5lk - 1</td>
<td>(-A_{ij,k}^{0,0} \cdot B_{ij,j}^{0,1} \cdot D_m/a^2)</td>
</tr>
<tr>
<td>5ji - 3</td>
<td>5lk</td>
<td>(L_x \cdot L_f \cdot A_{ij,k}^{0,0} \cdot B_{ij,j}^{0,0} \cdot G_m/a - L_f \cdot L_x \cdot A_{ij,k}^{1,0} \cdot B_{ij,j}^{0,0} \cdot D_m/a)</td>
</tr>
<tr>
<td>5ji - 2</td>
<td>5lk - 4</td>
<td>(-L_f \cdot A_{ij,k}^{0,0} \cdot B_{ij,j}^{0,0} \cdot \nu \cdot C/a)</td>
</tr>
<tr>
<td>5ji - 2</td>
<td>5lk - 3</td>
<td>(-L_x \cdot A_{ij,k}^{0,0} \cdot B_{ij,j}^{0,0} \cdot C/a^2 + L_x \cdot A_{ij,k}^{0,0} \cdot B_{ij,j}^{0,0} \cdot G_m/a^2)</td>
</tr>
<tr>
<td>5ji - 2</td>
<td>5lk - 2</td>
<td>([L_x \cdot L_f \cdot A_{ij,k}^{0,0} \cdot B_{ij,j}^{0,0} \cdot C/a^2 + L_f \cdot L_x \cdot A_{ij,k}^{0,0} \cdot B_{ij,j}^{1,1} \cdot G_m + L_x \cdot L_f \cdot A_{ij,k}^{0,0} \cdot B_{ij,j}^{0,0} \cdot G_m]/a^2)</td>
</tr>
<tr>
<td>5ji - 2</td>
<td>5lk - 1</td>
<td>(L_f \cdot A_{ij,k}^{0,0} \cdot B_{ij,j}^{0,0} \cdot G_m)</td>
</tr>
</tbody>
</table>
The non-zero coefficients of the mass matrix $\mathbf{M}$ are

<table>
<thead>
<tr>
<th>$m$</th>
<th>$n$</th>
<th>$\mathbf{M}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$5 ji - 4$</td>
<td>$5 lk - 4$</td>
<td>$L_s \cdot A_{i,k}^{0,0} \cdot B_{j,l}^{0,0} \cdot M_m$</td>
</tr>
<tr>
<td>$5 ji - 3$</td>
<td>$5 lk - 3$</td>
<td>$L_s \cdot L_f \cdot A_{i,k}^{0,0} \cdot B_{j,l}^{0,0} \cdot M_m$</td>
</tr>
<tr>
<td>$5 ji - 2$</td>
<td>$5 lk - 2$</td>
<td>$L_s \cdot L_f \cdot A_{i,k}^{0,0} \cdot B_{j,l}^{0,0} \cdot M_m$</td>
</tr>
<tr>
<td>$5 ji - 1$</td>
<td>$5 lk - 1$</td>
<td>$L_s \cdot L_f \cdot A_{i,k}^{0,0} \cdot B_{j,l}^{0,0} \cdot M_m$</td>
</tr>
<tr>
<td>$5 ji$</td>
<td>$5 lk$</td>
<td>$L_s \cdot A_{i,k}^{0,0} \cdot B_{j,l}^{0,0} \cdot M_m$</td>
</tr>
</tbody>
</table>
The non-zero coefficients of the geometric stiffness matrix $G_{cy}^e$ due to axially compressed forces are

<table>
<thead>
<tr>
<th>$m$</th>
<th>$n$</th>
<th>$G_{cy}^e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 $ji - 4$</td>
<td>5 $lk - 4$</td>
<td>$L_f / L_s \cdot A_{i,k}^{1,1} \cdot B_{j,l}^{0,0}$</td>
</tr>
<tr>
<td>5 $ji - 3$</td>
<td>5 $lk - 3$</td>
<td>$L_f / L_s \cdot A_{i,k}^{1,1} \cdot B_{j,l}^{0,0}$</td>
</tr>
<tr>
<td>5 $ji - 2$</td>
<td>5 $lk - 2$</td>
<td>$L_f / L_s \cdot A_{i,k}^{1,1} \cdot B_{j,l}^{0,0}$</td>
</tr>
</tbody>
</table>

where $C = Et / (1 - \nu^2)$, $D = Et / [12(1 - \nu^2)]$, $G_a = E / [2(1 + \nu)]$, $C_m = (1 - \nu) C / 2$, $G_m = G_a k t$, $D_m = (1 - \nu) D / 2$, and $E$, $\nu$, $k$ and $t$ are the Young’s modulus, Poisson’s ratio, shear correction factor and the thickness of the shell panels, and $M_a = \rho t$, $M_s = \rho t^3 / 12$.

The integrations are

$A_{i,k}^{\alpha,\beta} = \int_0^1 \bar{T}_i^a \cdot \bar{T}_k^\beta \, d\xi$,  
$B_{j,l}^{\alpha,\beta} = \int_0^1 \bar{T}_j^a \cdot \bar{T}_l^\beta \, d\eta$,  

where $i, k = 1, 2, \cdots, p + 2$, $j, l = 1, 2, \cdots, p + 2$, and the superscripts $\alpha$ and $\beta$ ($\alpha$, $\beta = 0, 1$) denote the order of the derivatives.
APPENDIX 7: MATRIX ENTRIES FOR CONICAL SHELL PANELS

Defined \( ji = j + (i-1)(p+2), \ lk = l + (k-1)(p+2). \)

The non-zero coefficients of the stiffness matrix \( K^e_{oo} \) are

<table>
<thead>
<tr>
<th>( m )</th>
<th>( n )</th>
<th>( K^e_{oo} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 ( ji - 4 )</td>
<td>5( lk - 4 )</td>
<td>[ L_f \left[ L_s \cdot sf \cdot D_{i,k}^{1,1} \cdot B_{j,l}^{0,0} \cdot C + L_f \cdot sf \cdot A_{i,k}^{0,1} \cdot B_{j,l}^{0,0} \cdot v \cdot C \right] + L_s \left[ L_f \cdot sf \cdot C_{i,k}^{0,0} \cdot B_{j,l}^{0,0} \cdot C \right] + L_s \left[ L_f \cdot sf \cdot C_{i,k}^{0,0} \cdot B_{j,l}^{1,1} \cdot C_m / sf \right] ]</td>
</tr>
</tbody>
</table>
| 5 \( ji - 4 \) | 5\( lk - 3 \) | \[ A_{i,k}^{0,1} \cdot B_{j,l}^{0,1} \cdot v \cdot C + L_s \cdot C_{i,k}^{0,0} \cdot B_{j,l}^{0,1} \cdot C \] + \[ A_{i,k}^{0,1} \cdot B_{j,l}^{0,1} \cdot C_m - L_s \cdot C_{i,k}^{0,0} \cdot B_{j,l}^{0,1} \cdot C_m \]
| 5 \( ji - 4 \) | 5\( lk - 2 \) | \[-L_f \cdot A_{i,k}^{0,1} \cdot B_{j,l}^{0,0} \cdot sf / tf \cdot v \cdot C - L_s \cdot L_f \cdot sf / tf \cdot C_{i,k}^{0,0} \cdot B_{j,l}^{0,0} \cdot C \]
| 5 \( ji - 3 \) | 5\( lk - 4 \) | \[ A_{i,k}^{0,1} \cdot B_{j,l}^{1,0} \cdot v \cdot C + L_s \cdot C_{i,k}^{0,0} \cdot B_{j,l}^{1,0} \cdot C \] + \[ C_{m} \cdot A_{i,k}^{0,1} \cdot B_{j,l}^{1,0} - L_s \cdot C_{i,k}^{0,0} \cdot B_{j,l}^{1,0} \cdot C_m \]
| 5 \( ji - 3 \) | 5\( lk - 3 \) | \[ L_s \left[ L_f \cdot C_{i,k}^{0,0} \cdot B_{j,l}^{1,1} \cdot C / sf + L_f \left[ L_s \cdot sf \cdot C_{m} \cdot D_{i,k}^{1,1} \cdot B_{j,l}^{0,0} \right] - L_f \cdot sf \cdot C_{m} \cdot A_{i,k}^{0,1} \cdot B_{j,l}^{0,0} - L_f \cdot sf \cdot A_{i,k}^{0,1} \cdot B_{j,l}^{0,0} \right] \] + \[ L_s \cdot L_f \cdot sf \cdot C_{m} \cdot C_{i,k}^{0,0} \cdot B_{j,l}^{0,0} \cdot C_m + L_s \cdot L_f \cdot sf / tf^2 \cdot C_{i,k}^{0,0} \cdot B_{j,l}^{0,0} \cdot G_m \] + \[ L_f / L_s \cdot sf / tf^2 \cdot C_{i,k}^{1,1} \cdot B_{j,l}^{1,0} \cdot D_m \]
| 5 \( ji - 3 \) | 5\( lk - 2 \) | \[-L_s \cdot C_{i,k}^{0,0} \cdot B_{j,l}^{1,0} \cdot C / tf + L_s / tf \cdot C_{i,k}^{0,0} \cdot B_{j,l}^{1,0} \cdot G_m \]
| 5 \( ji - 3 \) | 5\( lk - 1 \) | \[-C_{i,k}^{0,0} \cdot B_{j,l}^{1,0} \cdot D_m / tf \] |
| 5 \( ji - 3 \) | 5\( lk \) | \[ L_s \cdot L_f \cdot sf / tf \cdot A_{i,k}^{0,0} \cdot B_{j,l}^{0,0} \cdot G_m + L_f \cdot sf \cdot C_{i,k}^{0,0} \cdot B_{j,l}^{0,0} \cdot D_m / tf \] - \[ L_f / L_s \cdot sf \cdot A_{i,k}^{0,1} \cdot B_{j,l}^{0,0} \cdot D_m / tf \] |
| 5 \( ji - 2 \) | 5\( lk - 4 \) | \[-L_f \cdot sf / tf \cdot A_{i,k}^{0,1} \cdot B_{j,l}^{0,0} \cdot v \cdot C - L_s \cdot L_f \cdot sf / tf \cdot C_{i,k}^{0,0} \cdot B_{j,l}^{0,0} \cdot C \]
\[
\begin{align*}
5ji - 2 & \quad 5lk - 3 & \quad -L_s/L_j \cdot C_{ik}^{0,0} \cdot B_{jj}^{0,0} \cdot C + L_s \cdot C_{ik}^{0,0} \cdot B_{jj}^{0,0} \cdot G_m/|f| \\
5ji - 2 & \quad 5lk - 2 & \quad L_s \cdot L_j \cdot sf/|f|^2 \cdot C_{ik}^{0,0} \cdot B_{jj}^{0,0} \cdot C + L_s/L_j \cdot B_{jj}^{1,1} \cdot B_{jj}^{0,0} \cdot G_m + L_s/L_f \cdot C_{ik}^{0,0} \cdot B_{jj}^{1,1} \cdot G_m/|sf| \\
5ji - 2 & \quad 5lk - 1 & \quad L_f \cdot sf \cdot C_{ik}^{0,0} \cdot B_{jj}^{0,0} \cdot G_m \\
5ji - 2 & \quad 5lk & \quad L_s \cdot A_{ik}^{0,0} \cdot B_{jj}^{0,0} \cdot G_m \\
5ji - 1 & \quad 5lk - 3 & \quad -C_{ik}^{0,1} \cdot B_{jj}^{0,0} \cdot D_m/|f| \\
5ji - 1 & \quad 5lk - 2 & \quad L_f \cdot sf \cdot D_{ik}^{0,1} \cdot B_{jj}^{0,0} \cdot G_m \\
5ji - 1 & \quad 5lk - 1 & \quad L_f \cdot sf \cdot A_{ik}^{1,0} \cdot B_{jj}^{1,0} \cdot \nu \cdot D + L_s \cdot sf \cdot A_{ik}^{0,0} \cdot B_{jj}^{0,0} \cdot \nu \cdot D + L_s \cdot L_f \cdot sf \cdot C_{ik}^{0,0} \cdot B_{jj}^{1,0} \cdot D_m + L_s/L_f \cdot C_{ik}^{0,0} \cdot B_{jj}^{1,1} \cdot D_m/|sf| \\
5ji - 1 & \quad 5lk & \quad A_{ik}^{1,0} \cdot B_{jj}^{0,1} \cdot \nu \cdot D + L_s \cdot C_{ik}^{0,0} \cdot B_{jj}^{0,1} \cdot D - L_s \cdot C_{ik}^{0,0} \cdot B_{jj}^{1,0} \cdot D_m + A_{ik}^{0,1} \cdot B_{jj}^{1,0} \cdot D_m \\
5ji & \quad 5lk - 3 & \quad L_s \cdot L_f \cdot sf \cdot A_{ik}^{0,0} \cdot B_{jj}^{0,0} \cdot G_m/|f| + L_f \cdot sf \cdot D_m/|f| \cdot C_{ik}^{0,1} \cdot B_{jj}^{0,0} + L_s/L_f \cdot D_m/|f| \cdot A_{ik}^{0,1} \cdot B_{jj}^{0,0} \\
5ji & \quad 5lk - 2 & \quad L_s \cdot A_{ik}^{0,0} \cdot B_{jj}^{0,1} \cdot G_m \\
5ji & \quad 5lk - 1 & \quad A_{ik}^{0,1} \cdot B_{jj}^{0,1} \cdot \nu \cdot D + L_s \cdot C_{ik}^{0,0} \cdot B_{jj}^{0,0} \cdot D - L_s \cdot D_m \cdot C_{ik}^{0,0} \cdot B_{jj}^{0,1} \cdot D_m + A_{ik}^{0,1} \cdot B_{jj}^{1,0} \\
5ji & \quad 5lk & \quad L_s \cdot L_f \cdot sf \cdot D_{ik}^{0,0} \cdot B_{jj}^{0,0} \cdot G_m + L_s/L_f \cdot C_{ik}^{0,0} \cdot B_{jj}^{1,1} \cdot D/|sf| + L_s/L_f \cdot D_m \cdot A_{ik}^{0,1} \cdot B_{jj}^{0,0} - L_f \cdot sf \cdot D_m \cdot A_{ik}^{0,1} \cdot B_{jj}^{0,0} - L_f \cdot sf \cdot D_m \cdot A_{ik}^{0,1} \cdot B_{jj}^{0,0} + L_s/L_f \cdot D_m \cdot A_{ik}^{1,0} \cdot B_{jj}^{0,0} + L_f \cdot sf \cdot D_m \cdot A_{ik}^{1,0} \cdot B_{jj}^{0,0} + L_s/L_f \cdot D_m \cdot D_{ik}^{0,1} \cdot B_{jj}^{0,0}
\end{align*}
\]
The non-zero coefficients of the mass matrix \( M_{eo} \) are

<table>
<thead>
<tr>
<th>( m )</th>
<th>( n )</th>
<th>( M_{eo} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 5ji-4 )</td>
<td>( 5lk-4 )</td>
<td>( L_a \cdot L_f \cdot sf \cdot D_{i,k}^{0,0} \cdot B_{j,l}^{0,0} \cdot M_a )</td>
</tr>
<tr>
<td>( 5ji-3 )</td>
<td>( 5lk-3 )</td>
<td>( L_a \cdot L_f \cdot sf \cdot D_{i,k}^{0,0} \cdot B_{j,l}^{0,0} \cdot M_a )</td>
</tr>
<tr>
<td>( 5ji-2 )</td>
<td>( 5lk-2 )</td>
<td>( L_a \cdot L_f \cdot sf \cdot D_{i,k}^{0,0} \cdot B_{j,l}^{0,0} \cdot M_a )</td>
</tr>
<tr>
<td>( 5ji-1 )</td>
<td>( 5lk-1 )</td>
<td>( L_a \cdot L_f \cdot sf \cdot D_{i,k}^{0,0} \cdot B_{j,l}^{0,0} \cdot M_b )</td>
</tr>
<tr>
<td>( 5ji )</td>
<td>( 5lk )</td>
<td>( L_a \cdot L_f \cdot sf \cdot D_{i,k}^{0,0} \cdot B_{j,l}^{0,0} \cdot M_b )</td>
</tr>
</tbody>
</table>

The integrations are

\[
A_{i,k}^{\alpha,\beta} = \int_0^1 \tilde{f}_j^\alpha \cdot \tilde{f}_k^\beta \, d\xi,
\]

\[
B_{j,l}^{\alpha,\beta} = \int_0^1 \tilde{f}_j^\alpha \cdot \tilde{f}_l^\beta \, d\eta,
\]

\[
C_{i,k}^{\alpha,\beta} = \int_0^1 \frac{1}{(s_0 + L_x \xi)} \tilde{f}_j^\alpha \cdot \tilde{f}_k^\beta \, d\xi,
\]

\[
D_{i,k}^{\alpha,\beta} = \int_0^1 (s_0 + L_x \xi) \tilde{f}_j^\alpha \cdot \tilde{f}_k^\beta \, d\xi,
\]

where \( i, k = 1, 2, \ldots, p + 2 \), \( j, l = 1, 2, \ldots, p + 2 \), and the superscripts \( \alpha \) and \( \beta \) (\( \alpha \), \( \beta = 0, 1 \)) denote the order of the derivatives.
APPENDIX 8: MATRIX ENTRIES FOR SPHERICAL SHELL PANELS

Defined \( ji = j + (i-1)(p+2) \), \(lk = l + (k-1)(p+2)\).

The non-zero coefficients of the stiffness matrix \( K_{sp}^e \) are

<table>
<thead>
<tr>
<th>( m )</th>
<th>( n )</th>
<th>( K_{sp}^e )</th>
</tr>
</thead>
</table>
| 5 ji – 4 | 5lk – 4 | \( L_f/L_s \cdot A_{i,k}^{1,0} \cdot E_{j,l}^{0,0} \cdot C + L_s/L_f \cdot C_m \cdot A_{i,k}^{0,0} \cdot D_{j,l}^{1,1} \)  
\(-L_s \cdot C_m \cdot A_{i,k}^{0,0} \cdot E_{j,l}^{1,0} \cdot -L_s \cdot C_m \cdot A_{i,k}^{0,0} \cdot E_{j,l}^{0,0} \)  
\(+L_s/L_f \cdot C_m \cdot A_{i,k}^{0,0} \cdot F_{j,l}^{0,0} \cdot +L_s \cdot C_m \cdot A_{i,k}^{0,0} \cdot D_{j,l}^{0,0} \cdot G_m \)  
\(+L_s/L_f \cdot A_{i,k}^{0,0} \cdot D_{j,l}^{1,1} \cdot D_m/a^2 -L_s \cdot A_{i,k}^{0,0} \cdot E_{j,l}^{1,0} \cdot D_m/a^2 \)  
\(-L_s \cdot A_{i,k}^{0,0} \cdot E_{j,l}^{1,0} \cdot D_m/a^2 \) |

| 5 ji – 4 | 5lk – 3 | \( L_f \cdot A_{i,k}^{1,0} \cdot G_{j,l}^{0,0} \cdot C + A_{i,k}^{1,0} \cdot B_{j,l}^{0,1} \cdot v \cdot C \) |

| 5 ji – 4 | 5lk – 2 | \( -L_f \cdot A_{i,k}^{1,0} \cdot B_{j,l}^{0,0} \cdot C \) |

| 5 ji – 4 | 5lk – 1 | \( -L_s/L_f \cdot A_{i,k}^{1,0} \cdot D_{j,l}^{1,1} \cdot D_m/a -L_s \cdot L_f \cdot A_{i,k}^{0,0} \cdot F_{j,l}^{0,0} \cdot D_m/a \) |

| 5 ji – 4 | 5lk | \(-A_{i,k}^{0,1} \cdot B_{j,l}^{0,0} \cdot D_m/a + L_f \cdot A_{i,k}^{0,1} \cdot G_{j,l}^{0,0} \cdot D_m/a \) |

| 5 ji – 3 | 5lk – 4 | \( L_f \cdot A_{i,k}^{0,1} \cdot G_{j,l}^{0,0} \cdot C + A_{i,k}^{0,1} \cdot B_{j,l}^{0,1} \cdot v \cdot C \) |

| 5 ji – 3 | 5lk | \(+A_{i,k}^{0,1} \cdot B_{j,l}^{0,0} \cdot C_m -L_f \cdot A_{i,k}^{0,1} \cdot D_{j,l}^{0,0} \cdot C_m \)  
\(+A_{i,k}^{0,1} \cdot B_{j,l}^{0,0} \cdot D_m/a^2 -L_f \cdot A_{i,k}^{0,1} \cdot G_{j,l}^{0,0} \cdot D_m/a^2 \) |
\[ L_s \cdot L_f \cdot A_{i,k}^{0,0} \cdot F_{j,j}^{0,0} \cdot C + L_s \cdot A_{i,k}^{0,0} \cdot E_{j,j}^{1,0} \cdot \mathbf{v} \cdot C \\
+ L_s \cdot A_{i,k}^{0,0} \cdot E_{j,j}^{0,1} \cdot \mathbf{v} \cdot C + L_s / L_f \cdot A_{i,k}^{0,0} \cdot D_{j,j}^{1,0} \cdot C + L_s / L_f \cdot A_{i,k}^{0,0} \cdot D_{j,j}^{1,0} \cdot G_m \\
+ L_f / L_s \cdot A_{i,k}^{1,0} \cdot C_{j,j}^{0,0} \cdot \mathbf{a}^2 \]

\[ -L_s \cdot L_f \cdot A_{i,k}^{0,0} \cdot E_{j,j}^{0,0} \cdot C - L_s \cdot A_{i,k}^{0,0} \cdot D_{j,j}^{1,0} \cdot \mathbf{v} \cdot C \\
\]

\[ L_f \cdot A_{i,k}^{0,0} \cdot G_{j,j}^{0,0} \cdot D_m / a - A_{i,k}^{0,0} \cdot B_{j,j}^{0,1} \cdot D_m / a \\
\]

\[ L_s \cdot L_f \cdot a \cdot A_{i,k}^{0,0} \cdot D_{j,j}^{0,0} \cdot G_m - L_f / L_s \cdot A_{i,k}^{1,0} \cdot C_{j,j}^{0,0} \cdot D_m / a \\
\]

\[ -L_s \cdot L_f \cdot A_{i,k}^{0,1} \cdot B_{j,j}^{0,0} \cdot C - L_s \cdot A_{i,k}^{0,1} \cdot B_{j,j}^{0,0} \cdot \mathbf{v} \cdot C \\
+ L_f / L_s \cdot A_{i,k}^{1,0} \cdot B_{j,j}^{0,0} \cdot G_m \\
\]

\[ -L_s \cdot L_f \cdot A_{i,k}^{0,0} \cdot E_{j,j}^{0,0} \cdot C - L_s \cdot L_f \cdot A_{i,k}^{0,0} \cdot E_{j,j}^{0,0} \cdot \mathbf{v} \cdot C \\
\]

\[ L_s \cdot A_{i,k}^{0,0} \cdot D_{j,j}^{1,0} \cdot C - L_s \cdot A_{i,k}^{0,0} \cdot D_{j,j}^{1,0} \cdot \mathbf{v} \cdot C \\
+ L_s \cdot A_{i,k}^{0,0} \cdot D_{j,j}^{1,0} \cdot G_m \\
\]

\[ 2 \cdot L_s \cdot L_f \cdot A_{i,k}^{0,0} \cdot D_{j,j}^{0,0} \cdot C + 2 \cdot L_s \cdot L_f \cdot A_{i,k}^{0,0} \cdot D_{j,j}^{0,0} \cdot \mathbf{v} \cdot C \\
+ L_f / L_s \cdot A_{i,k}^{1,0} \cdot C_{j,j}^{0,0} \cdot G_m + L_s / L_f \cdot A_{i,k}^{0,0} \cdot C_{j,j}^{1,0} \cdot G_m \\
\]

\[ L_f \cdot A_{i,k}^{0,0} \cdot B_{j,j}^{0,0} \cdot G_m \\
\]

\[ L_s \cdot a \cdot A_{i,k}^{0,0} \cdot D_{j,j}^{0,0} \cdot G_m \\
\]

\[ L_s \cdot L_f \cdot a \cdot A_{i,k}^{0,0} \cdot D_{j,j}^{0,0} \cdot G_m + L_s \cdot A_{i,k}^{0,0} \cdot E_{j,j}^{1,0} \cdot D_m / a \\
\]

\[ -L_s / L_f \cdot A_{i,k}^{0,0} \cdot D_{j,j}^{1,0} \cdot D_m / a - L_s / L_f \cdot A_{i,k}^{0,0} \cdot F_{j,j}^{0,0} \cdot D_m / a \\
+ L_s \cdot A_{i,k}^{0,0} \cdot E_{j,j}^{0,1} \cdot D_m / a \\
\]

\[ L_f \cdot D_m / a \cdot A_{i,k}^{0,1} \cdot G_{j,j}^{0,0} - A_{i,k}^{0,1} \cdot B_{j,j}^{1,0} \cdot D_m / a \\
\]

\[ L_f \cdot a \cdot A_{i,k}^{0,1} \cdot B_{j,j}^{0,0} \cdot G_m \\
\]

\[ L_s \cdot L_f \cdot a^2 \cdot A_{i,k}^{0,0} \cdot D_{j,j}^{0,0} \cdot G_m + L_s / L_f \cdot A_{i,k}^{1,0} \cdot C_{j,j}^{0,0} \cdot D \\
\]

\[ L_f \cdot L_f \cdot D_m \cdot A_{i,k}^{0,0} \cdot F_{j,j}^{0,0} - L_s \cdot D_m \cdot A_{i,k}^{0,0} \cdot E_{j,j}^{0,1} \\
- L_s \cdot D_m \cdot A_{i,k}^{0,0} \cdot E_{j,j}^{0,0} + L_s / L_f \cdot D_m \cdot A_{i,k}^{0,0} \cdot D_{j,j}^{1,0} \]
The non-zero coefficients of the mass matrix $M^e_{ij}$ are

<table>
<thead>
<tr>
<th>$m$</th>
<th>$n$</th>
<th>$M^e_{ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$5ji - 4$</td>
<td>$5lk - 4$</td>
<td>$L_s \cdot L_f \cdot a^2 \cdot A^0_{i,k} \cdot D_{j,j}^0 \cdot M_a$</td>
</tr>
<tr>
<td>$5ji - 3$</td>
<td>$5lk - 3$</td>
<td>$L_s \cdot L_f \cdot a^2 \cdot A^0_{i,k} \cdot D_{j,j}^0 \cdot M_a$</td>
</tr>
<tr>
<td>$5ji - 2$</td>
<td>$5lk - 2$</td>
<td>$L_s \cdot L_f \cdot a^2 \cdot A^0_{i,k} \cdot D_{j,j}^0 \cdot M_a$</td>
</tr>
<tr>
<td>$5ji - 1$</td>
<td>$5lk - 1$</td>
<td>$L_s \cdot L_f \cdot a^2 \cdot A^0_{i,k} \cdot D_{j,j}^0 \cdot M_a$</td>
</tr>
<tr>
<td>$5ji$</td>
<td>$5lk$</td>
<td>$L_s \cdot L_f \cdot a^2 \cdot A^0_{i,k} \cdot D_{j,j}^0 \cdot M_b$</td>
</tr>
</tbody>
</table>
The integrations are

\[ A_{i,k}^{\alpha,\beta} = \int_0^1 \bar{f}_i^a \cdot \bar{f}_k^\beta \, d\xi, \]
\[ B_{j,l}^{\alpha,\beta} = \int_0^1 \bar{f}_j^a \cdot \bar{f}_l^\beta \, d\eta, \]
\[ C_{i,k}^{\alpha,\beta} = \int_0^1 \frac{1}{\sin(f_0 + L_j \eta)} \bar{f}_i^a \cdot \bar{f}_k^\beta \, d\eta, \]
\[ D_{i,k}^{\alpha,\beta} = \int_0^1 \sin(f_0 + L_j \eta) \bar{f}_i^a \cdot \bar{f}_k^\beta \, d\eta, \]
\[ E_{i,k}^{\alpha,\beta} = \int_0^1 \cos(f_0 + L_j \eta) \bar{f}_i^a \cdot \bar{f}_k^\beta \, d\eta, \]
\[ F_{i,k}^{\alpha,\beta} = \int_0^1 \left[ \frac{\cos(f_0 + L_j \eta)}{\sin(f_0 + L_j \eta)} \right]^2 \bar{f}_i^a \cdot \bar{f}_k^\beta \, d\eta, \]
\[ G_{i,k}^{\alpha,\beta} = \int_0^1 \frac{\cos(f_0 + L_j \eta)}{\sin(f_0 + L_j \eta)} \bar{f}_i^a \cdot \bar{f}_k^\beta \, d\eta, \]

where \( i, k = 1, 2, \ldots, p + 2, \quad j, l = 1, 2, \ldots, p + 2, \) and the superscripts \( \alpha \) and \( \beta \) (\( \alpha, \beta = 0, 1 \)) denote the order of the derivatives.
VITA

Name: FAN Jie
Place of birth: Changsha, Hunan Province, P. R. China
Date of birth: 25 October, 1981
E-mail:

POST-SECONDARY EDUCATION AND DEGREES

Huazhong University of Science and Technology
Wuhan, 430074, P. R. China

Huazhong University of Science and Technology
Wuhan, 430074, P. R. China

City University of Hong Kong
Hong Kong, P. R. China
June 2006—September 2010, Ph. D.
LIST OF PUBLICATIONS

Journals:


Conferences:


