UNIFORM ASYMPTOTICS OF THE MEIXNER POLYNOMIALS AND SOME q-ORTHOGONAL POLYNOMIALS

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Uniform Asymptotics of the Meixner Polynomials and Some q-Orthogonal Polynomials
關於 Meixner 多項式和一些 q 正交多項式的一致漸近分析

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Abstract

In this thesis, we study the uniform asymptotic behavior of the Meixner polynomials and some q-orthogonal polynomials as the polynomial degree n tends to infinity.

Using the steepest descent method of Deift-Zhou, we derive uniform asymptotic formulas for the Meixner polynomials. These include an asymptotic formula in a neighborhood of the origin, a result which as far as we are aware has not yet been obtained previously. This particular formula involves a special function, which is the uniformly bounded solution to a scalar Riemann-Hilbert problem, and which is asymptotically (as $n \to \infty$) equal to the constant "1" except at the origin. Numerical computation by using our formulas, and comparison with earlier results, are also given.

With some modifications of Laplace's approximation, we obtain uniform asymptotic formulas for the Stieltjes-Wigert polynomial, the q^{-1}-Hermite polynomial and the q-Laguerre polynomial. In these formulas, the q-Airy polynomial, defined by truncating the q-Airy function, plays a significant role. While the standard Airy function, used frequently in the uniform asymptotic formulas for classical orthogonal polynomials, behaves like the exponential function on one side and the trigonometric functions on the other side of an extreme zero, the q-Airy polynomial behaves like the q-Airy function on one side and the q-Theta function on the other side. The last two special functions are involved in the local asymptotic formulas of the q-orthogonal polynomials. It seems therefore reasonable to expect that the q-Airy polynomial will play an important role in the asymptotic theory of the q-orthogonal polynomials.

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Chapter 1

Introduction

1.1 The Meixner polynomials and some q-orthogonal polynomials

In this thesis, we investigate the asymptotic behavior of the Meixner polynomials and some q-orthogonal polynomials. These polynomials have many applications in statistical physics. For instance, the Meixner polynomials are related to the study of the shape fluctuations in a certain two dimensional random growth model; see [18] and references therein. Furthermore, it has been shown in [4] that the Stieltjes-Wigert polynomials, one typical example of the q-orthogonal polynomials, are of significance in the study of non-intersecting random walks.

For $\beta > 0$ and $0 < c < 1$, the Meixner polynomials are explicitly given by [19, (1.9.1)]

$$M_n(z;\beta,c) = {}_2F_1\left(\begin{matrix}-n,-z\\ \beta\end{matrix}\middle|1-\frac{1}{c}\right) = \sum_{k=0}^{n}\frac{(-n)_k(-z)_k}{(\beta)_k k!}\left(1-\frac{1}{c}\right)^k. \tag{1.1}$$

They satisfy the discrete orthogonality condition [19, (1.9.2)]

$$\sum_{k=0}^{\infty}\frac{c^k(\beta)_k}{k!}M_m(k;\beta,c)M_n(k;\beta,c) = \frac{c^{-n}n!}{(\beta)_n(1-c)^{\beta}}\delta_{mn}, \tag{1.2}$$

and the recurrence relation [19, (1.9.3)]

$$\begin{aligned} zM_n(z;\beta,c) =& \frac{(n+\beta)c}{c-1}M_{n+1}(z;\beta,c) - \frac{n+(n+\beta)c}{c-1}M_n(z;\beta,c) \\ &+\frac{n}{c-1}M_{n-1}(z;\beta,c). \end{aligned} \tag{1.3}$$

Not much is known about the asymptotic behavior of the Meixner polynomials for large values of n. Using probabilistic arguments, Maejima and Van

Assche [21] have given an asymptotic formula for $M_n(n\alpha; \beta, c)$ when $\alpha < 0$ and β is a positive integer. Their result is in terms of elementary functions. In [16], Jin and Wong have used the steepest-descent method for integrals to derive two infinite asymptotic expansions for $M_n(n\alpha; \beta, c)$. One holds uniformly for $0 < \varepsilon \le \alpha \le 1 + \varepsilon$, and the other holds uniformly for $1 - \varepsilon \le \alpha \le M < \infty$; both expansions involve the parabolic cylinder function and its derivative.

In view of Gauss's contiguous relations for hypergeometric functions ([1, Section 15.2]), we may restrict our study to the case $1 \le \beta < 2$. Fixing any $0 < c < 1$ and $1 \le \beta < 2$, we intend to investigate the large-n behavior of $M_n(nz - \beta/2; \beta, c)$ for z in the whole complex plane. Our approach is based on the nonlinear steepest-descent method for oscillatory Riemann-Hilbert problems, first introduced by Deift and Zhou [8] for nonlinear partial differential equations, and later developed in [7] and [2, 3] for orthogonal polynomials with respect to exponential weights or a general class of discrete weights. As in [2,3], our formulas are given in several different regions. One may decompose the complex plane into less regions and obtain some global asymptotic results as in [6]. But, here we are only able to find uniform asymptotic formulas in several local regions; see Theorem 2.21 below.

To study the q-orthogonal polynomials we shall introduce some notations. For $q \in (0, 1)$, define

$$(a; q)_0 := 1, \qquad (a; q)_n := \prod_{k=1}^{n}(1 - aq^{k-1}), \qquad n = 1, 2, \cdots; \tag{1.4}$$

this definition remains valid when n is infinite; see [12, p. 7]. In terms of these notations, the q^{-1}-Hermite polynomials $h_n(z|q)$ with $z = \sinh\xi$, the Stieltjes-Wigert polynomials $S_n(z; q)$, and the q-Laguerre polynomials $L_n^{\alpha}(z; q)$ are defined by

$$h_n(\sinh\xi|q) := \sum_{k=0}^{n} \frac{(q^{n-k+1};q)_k}{(q;q)_k} q^{k^2-kn}(-1)^k e^{(n-2k)\xi}, \tag{1.5}$$

$$S_n(z;q) := \sum_{k=0}^{n} \frac{q^{k^2}}{(q;q)_k (q;q)_{n-k}} (-z)^k, \tag{1.6}$$

$$L_n^{\alpha}(z;q) := \sum_{k=0}^{n} \frac{(q^{\alpha+k+1};q)_{n-k}}{(q;q)_k (q;q)_{n-k}} q^{k^2+\alpha k}(-z)^k; \tag{1.7}$$

see [13].

Unlike ordinary orthogonal polynomials, the q-orthogonal polynomials do not satisfy any second-order ordinary differential equation or have any integral representation. Therefore the powerful tools, such as the WKB method for differential equations and the steepest-descent method for integrals, are not applicable. Recently, Ismail and Zhang [15] intoduce a logarithmic scaling, namely $z = \sinh\xi := (q^{-nt}u - q^{nt}u^{-1})/2$ with $t \geq 0$ and $u \neq 0$, and derive three different asymptotic formulas for the q^{-1}-Hermite polynomials with respect to following three cases:

1) $t \geq 1/2$;

2) $0 \leq t < 1/2$ and $t \in \mathbb{Q}$;

3) $0 \leq t < 1/2$ and $t \notin \mathbb{Q}$.

Their formulas involve the q-Airy function [14, (2.9)]

$$A_q(z) := \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k} (-z)^k, \tag{1.8}$$

and the q-Theta function [27, p. 463]

$$\Theta_q(z) := \sum_{k=-\infty}^{\infty} q^{k^2} z^k. \tag{1.9}$$

Similar results are also obtained for the Stieltjes-Wigert polynomials and the q-Laguerre polynomials.

Note that their formula in the second case when t is rational is different from that in the third case when t is irrational. One may be curious to know if there exists a uniform asymptotic formula for all rational and irrational $t \in [0, 1/2)$. Also, one would like to ask if it is possible to find a uniform asymptotic formula when t is in a neighborhood of $1/2$. With the aid of discrete Laplace's approximation (cf. [26]), we are able to answer these two questions. By the same scaling as mentioned before, we derive two uniform asymptotic formulas for the q^{-1}-Hermite polynomials with respect to following two cases:

1) $0 \le t < 1/2$;

2) $t > 1/2 - \delta$ where $\delta > 0$ is a fixed small number.

It turns out that, in stead of the q-Airy function given in (1.8), our formulas involve the polynomial

$$A_{q,n}(z) := \sum_{k=0}^{n} \frac{q^{k^2}}{(q;q)_k}(-z)^k. \tag{1.10}$$

Since this is simply the n-th partial sum of the q-Airy function, we call it the *q-Airy polynomial.* Note that the chaotic behavior mentioned in [15] does not exist in our formulas since we have a unified asymptotic formula for all $0 \le t < 1/2$, whether t is rational or not. Similar improvements are also given to the results obtained by Ismail and Zhang for the Stieltjes-Wigert polynomials and the q-Laguerre polynomials. Here, we would like to mention that our method is most likely also applicable for other types of q-orthogonal polynomials.

1.2 Method of asymptotic analysis for the Meixner polynomials

In this section we give an outline of the procedure in asymptotic analysis for the Meixner polynomials. First, we use the orthogonality property (1.2) to relate the Meixner polynomials with a 2×2 matrix-valued function which is the unique solution to an interpolation problem. Our problem subsequently becomes studying the asymptotic behavior of this matrix-valued function. Second, we construct an equivalent Riemann-Hilbert problem, the solution of which can be written explicitly based on the solution to the basic interpolation problem. This is an invertible transformation and thus we only need to investigate the equivalent Riemann-Hilbert problem. With the aid of the equilibrium measure, we transform the Riemann-Hilbert problem into another oscillatory one. Finally, the Deift-Zhou steepest-descent method for oscillatory Riemann-Hilbert problem is applied and what remains is a study of a global Riemann-Hilbert problem which can be divided into several locally solvable problems. It should be noted that the solutions to these local problems are not unique. We must choose suitable solutions that are asymptotically equal to each other in the overlapped region. By piecing these solutions together, we build a function that is defined in the whole complex plane. This matrix-valued function is proved to be an approximate solution to the global Riemann-Hilbert problem. By tracing along the transformations back to the original interpolation problem, we obtain the asymptotic formulas for the Meixner polynomials.

1.3 Method of asymptotic analysis for some q-orthogonal polynomials

In this section we use a simple example to illustrate the idea of discrete Laplace's approximation, which will be used in asymptotic analysis for some q-orthogonal polynomials. Let $\phi(x)$ and $h(x)$ be two real-valued continuous functions defined in the finite interval $\alpha \leq x \leq \beta$. Assume that $h(x)$ has a single minimum in the interval, namely at $x = \alpha$, and that the infimum of $h(x)$ in any closed sub-interval not containing α is greater than $h(\alpha)$. Furthermore, assume that $h''(x)$ is continuous, $h'(\alpha) = 0$ and $h''(\alpha) > 0$. Then, Laplace's approximation states that the integral

$$I(\lambda) = \int_{\alpha}^{\beta} \phi(x) e^{-\lambda h(x)} dx \tag{1.11}$$

has the asymptotic formula

$$I(\lambda) \sim \phi(\alpha) e^{-\lambda h(\alpha)} \left[\frac{\pi}{2\lambda h''(\alpha)} \right]^{\frac{1}{2}} \tag{1.12}$$

as $\lambda \to +\infty$; see [5, p.39] or [28, p.57].

Now, put $\lambda = n^2$ and make the change of variable $x = \alpha + (\beta - \alpha)t$ so that the integral in (1.11) becomes

$$I(n^2) = (\beta - \alpha)\phi(\alpha) e^{-n^2 h(\alpha)} \int_0^1 f(t) e^{-n^2 g(t)} dt, \tag{1.13}$$

where $f(t) := \phi(x)/\phi(\alpha)$ and $g(t) := h(x) - h(\alpha)$. If we set $q := e^{-1}$, $k := nt$, $f_n(k) := \frac{1}{n} f(\frac{k}{n})$ and $g_n(k) := n^2 g(\frac{k}{n})$, then the integral in (1.13) can be written as

$$\int_0^1 f(t) e^{-n^2 g(t)} dt = \int_0^n f_n(k) q^{g_n(k)} dk.$$

A discrete form of the last integral is the finite sum

$$I_n(1|q) := \sum_{k=0}^{n} f_n(k) q^{g_n(k)}, \tag{1.14}$$

where $f_n(k)$ and $g_n(k)$ are two functions defined on nonnegative integers $\mathbb{N}$ and $q \in (0,1)$. We intend to investigate the behavior of the sum $I_n(1|q)$ as $n \to \infty$. As we shall see, its asymptotic behavior is given in terms of the q-Theta function (1.9)

$$\Theta_q(z) := \sum_{k=-\infty}^{\infty} q^{k^2} z^k, \qquad 0 < q < 1.$$

Note that $\Theta_q(1)$ is a continuous function of $q \in (0,1)$, since the infinite sum $\sum_{k=-\infty}^{\infty} q^{k^2}$ converges uniformly for q in any compact subset of $(0,1)$.

Theorem 1.1. *Assume that the following conditions hold:*

(i) $f_n(0) = 1$, $g_n(0) = 0$;

(ii) there exists a constant $M > 0$ such that $|f_n(k)| \leq M$ for $0 \leq k \leq n$;

(iii) for any $\delta \in (0,1)$ there exist a constant $A_\delta > 0$ and a positive integer $N(\delta)$ such that $g_n(k) \geq A_\delta n^2$ for all $n\delta \leq k \leq n$ and $n > N(\delta)$;

(iv) for some fixed $c_0 > 0$ and for any small $\varepsilon > 0$, there exist $\delta(\varepsilon) > 0$ and $N(\varepsilon) \in \mathbb{N}$ such that $|f_n(k) - 1| < \varepsilon$ and $|g_n(k) - c_0 k^2| \leq \varepsilon k^2$, whenever $0 \leq k \leq n\delta(\varepsilon)$ and $n > N(\varepsilon)$.

Then, we have

$$I_n(1|q) := \sum_{k=0}^{n} f_n(k) q^{g_n(k)} \sim \frac{1}{2}[\Theta_{\widetilde{q}}(1) + 1] \qquad \text{as} \quad n \to \infty, \tag{1.15}$$

where $\widetilde{q} := q^{c_0}$.

Proof. For any small $\varepsilon > 0$, we choose $\delta := \delta(\varepsilon)$ and $N(\varepsilon)$ as in (iv). Split the sum $I_n(1|q)$ into two so that $I_n(1|q) = I_1^* + I_2^*$, where

$$I_1^* := \sum_{k=0}^{\lfloor n\delta \rfloor} f_n(k) q^{g_n(k)}$$

and

$$I_2^* := \sum_{k=\lfloor n\delta \rfloor + 1}^{n} f_n(k) q^{g_n(k)}.$$

Simple estimation gives

$$I_1^* < \sum_{k=0}^{\lfloor n\delta \rfloor} (1+\varepsilon) q^{k^2(c_0 - \varepsilon)}$$

and

$$I_1^* > \sum_{k=0}^{\lfloor n\delta \rfloor} (1-\varepsilon) q^{k^2(c_0 + \varepsilon)},$$

from which we obtain

$$\frac{1-\varepsilon}{2}[\Theta_{q^{c_0+\varepsilon}}(1) + 1] \leq \varliminf_{n\to\infty} I_1^* \leq \varlimsup_{n\to\infty} I_1^* \leq \frac{1+\varepsilon}{2}[\Theta_{q^{c_0-\varepsilon}}(1) + 1].$$

By conditions (ii) and (iii), we also have

$$|I_2^*| \leq \sum_{k=\lfloor n\delta \rfloor + 1}^{n} M q^{n^2 A_\delta} \leq n M q^{n^2 A_\delta}.$$

Thus, $\lim_{n\to\infty} I_2^* = 0$ and

$$\frac{1-\varepsilon}{2}[\Theta_{q^{c_0+\varepsilon}}(1)+1] \le \varliminf_{n\to\infty} I_n(1|q) \le \varlimsup_{n\to\infty} I_n(1|q) \le \frac{1+\varepsilon}{2}[\Theta_{q^{c_0-\varepsilon}}(1)+1].$$

Since ε is arbitrary, the desired result (1.15) follows. $\square$

Chapter 2

Asymptotics of the Meixner Polynomials

2.1 The basic interpolation problem

From (1.1), the monic Meixner polynomials are given by

$$\pi_n(z) := (\beta)_n (1 - \frac{1}{c})^{-n} M_n(z; \beta, c). \tag{2.1}$$

On account of (1.3), we obtain the recurrence relation

$$z\pi_n(z) = \pi_{n+1}(z) + \frac{n + (n+\beta)c}{1-c}\pi_n(z) + \frac{n(n+\beta-1)c}{(1-c)^2}\pi_{n-1}(z). \tag{2.2}$$

The orthogonality property of $\pi_n(z)$ can be derived from (1.2), and we have

$$\sum_{k=0}^{\infty} \pi_m(k)\pi_n(k)w(k) = \delta_{mn}/\gamma_n^2, \tag{2.3}$$

where

$$\gamma_n^2 = \frac{(1-c)^{2n+\beta}c^{-n}}{\Gamma(n+\beta)\Gamma(n+1)} \tag{2.4}$$

and

$$w(z) := \frac{\Gamma(z+\beta)}{\Gamma(z+1)}c^z. \tag{2.5}$$

Let $P(z)$ be the 2×2 matrix defined by

$$P(z) := \begin{pmatrix} \pi_n(z) & \sum_{k=0}^{\infty} \dfrac{\pi_n(k)w(k)}{z-k} \\ \gamma_{n-1}^2 \pi_{n-1}(z) & \sum_{k=0}^{\infty} \dfrac{\gamma_{n-1}^2 \pi_{n-1}(k)w(k)}{z-k} \end{pmatrix}. \tag{2.6}$$

For consistency, we shall use capital letters to denote matrix-valued functions that depend on the large parameter n. Therefore, all the matrices P, Q, R, S, T, M and K depend on both z and n. The following proposition states that $P(z)$ is the unique solution to an interpolation problem, which is the discrete analogue of the Riemann-Hilbert problem corresponding to the orthogonal polynomials with continuous weights; see [10, 11].

Proposition 2.1. *The matrix-valued function $P(z)$ defined in (2.6) is the unique solution to the following interpolation problem:*

(P1) $P(z)$ is analytic in $\mathbb{C} \setminus \mathbb{N}$;

(P2) at each $z = k \in \mathbb{N}$, the first column of $P(z)$ is analytic and the second column of $P(z)$ has a simple pole with residue

$$\operatorname*{Res}_{z=k} P(z) = \lim_{z\to k} P(z) \begin{pmatrix} 0 & w(z) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & w(k)P_{11}(k) \\ 0 & w(k)P_{21}(k) \end{pmatrix}; \tag{2.7}$$

(P3) for z bounded away from $\mathbb{N}$, $P(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^{n} \end{pmatrix} = I + O(|z|^{-1})$ *as $z \to \infty$.*

Proof. Since $w(k)$ decays exponentially to zero as $k \to +\infty$, the summations in the second column of $P(z)$ in (2.6) are uniformly convergent for z in any compact subset of $\mathbb{C} \setminus \mathbb{N}$. Therefore, (P1) is obvious.

For each $k \in \mathbb{N}$, we have from (2.6)

$$\operatorname*{Res}_{z=k} P_{12}(z) = \pi_n(k)w(k) = P_{11}(k)w(k),$$

and

$$\operatorname*{Res}_{z=k} P_{22}(z) = \gamma_{n-1}^2 \pi_{n-1}(k)w(k) = P_{21}(k)w(k).$$

Thus, (P2) follows.

To prove (P3) we only need to show that $P_{12}(z)z^n = O(|z|^{-1})$ and $P_{22}(z)z^n = 1 + O(|z|^{-1})$ as $z \to \infty$ and for z bounded away from $\mathbb{N}$. Using the following expansion

$$\frac{1}{z-k}=\sum_{i=0}^{n-1}\frac{k^i}{z^{i+1}}+\frac{1}{z^{n+1}}\frac{k^n}{1-k/z},$$

we have

$$P_{12}(z)z^n=\sum_{i=0}^{n-1}z^{n-i-1}\sum_{k=0}^{\infty}k^i\pi_n(k)w(k)+\frac{1}{z}\sum_{k=0}^{\infty}\frac{k^n\pi_n(k)w(k)}{1-k/z}.$$

The orthogonality property (2.3) implies that $\sum_{k=0}^{\infty}k^i\pi_n(k)w(k)=0$ for any $i=0,1,\cdots,n-1$. Thus, we obtain

$$P_{12}(z)z^n=\frac{1}{z}\sum_{k=0}^{\infty}\frac{k^n\pi_n(k)w(k)}{1-k/z}.$$

Since z is bounded away from $\mathbb{N}$, it is easily seen that the last sum is uniformly bounded. Hence, we have $P_{12}(z)z^n=O(|z|^{-1})$ as $z\to\infty$. On the other hand, we also have

$$P_{22}(z)z^n=\sum_{i=0}^{n-1}z^{n-i-1}\gamma_{n-1}^2\sum_{k=0}^{\infty}k^i\pi_{n-1}(k)w(k)+\frac{1}{z}\sum_{k=0}^{\infty}\frac{\gamma_{n-1}^2k^n\pi_{n-1}(k)w(k)}{1-k/z}.$$

Again, using the orthogonality property (2.3), we obtain $\sum_{k=0}^{\infty}k^i\pi_{n-1}(k)w(k)=\delta_{i,n-1}/\gamma_{n-1}^2$ for any $i=0,1,\cdots,n-1$. Thus, it is readily seen that $P_{22}(z)z^n=1+O(|z|^{-1})$ as $z\to\infty$ and for z bounded away from $\mathbb{N}$. This ends our proof of (P3).

The uniqueness of the solution follows from Liouville's theorem. Indeed, condition (2.7) implies that the residue of $\det P(z)$ at $k\in\mathbb{N}$ is zero. Thus, the determinant function $\det P(z)$ can be analytically continued to an entire function. Condition (P3), together with Liouville's theorem, implies that $\det P(z)=1$. Therefore, $P(z)$ is invertible in $\mathbb{C}\setminus\mathbb{N}$. Let $\widetilde{P}(z)$ be a second solution to the interpolation problem (P1)-(P3). It is easily seen that the residue of $P(z)\widetilde{P}^{-1}(z)$ at $k\in\mathbb{N}$ is zero. Hence, $P(z)\widetilde{P}^{-1}(z)$ can be extended to an entire function. Again, using condition (P3), we obtain from Liouville's theorem that $P(z)\widetilde{P}^{-1}(z)=I$. This establishes the uniqueness. $\square$

2.2 The equivalent Riemann-Hilbert problem

In this section we first introduce two transformations $P \to Q$ and $Q \to R$. It will be shown that the matrix-valued function $R(z)$ is the unique solution to a Riemann-Hilbert problem. At the end of this section we make the third transformation $R \to S$ with the aid of the equilibrium measure.

The first transformation $P \to Q$ involves the following rescaling:

$$U(z) := n^{-n\sigma_3} P(nz - \beta/2) = \begin{pmatrix} n^{-n} & 0 \\ 0 & n^n \end{pmatrix} P(nz - \beta/2). \tag{2.8}$$

Here, $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a Pauli matrix. In this chapter, we will also make use of another Pauli matrix $\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; see (2.117). Let $\mathbb{X}$ denote the set defined by

$$\mathbb{X} := \{X_k\}_{k=0}^{\infty}, \qquad \text{where } X_k := \frac{k + \beta/2}{n}. \tag{2.9}$$

The X_k's are called *nodes*. Our first transformation is given by

$$\begin{aligned} Q(z) &:= U(z) \left[\prod_{j=0}^{n-1} (z - X_j) \right]^{-\sigma_3} = n^{-n\sigma_3} P(nz - \beta/2) \left[\prod_{j=0}^{n-1} (z - X_j) \right]^{-\sigma_3} \\ &= \begin{pmatrix} n^{-n} & 0 \\ 0 & n^n \end{pmatrix} P(nz - \beta/2) \begin{pmatrix} \prod_{j=0}^{n-1} (z - X_j)^{-1} & 0 \\ 0 & \prod_{j=0}^{n-1} (z - X_j) \end{pmatrix}, \end{aligned} \tag{2.10}$$

and the interpolation problem corresponding to $Q(z)$ is given below.

Proposition 2.2. *The matrix-valued function $Q(z)$ defined in (2.10) is the unique solution to the following interpolation problem:*

(Q1) $Q(z)$ is analytic in $\mathbb{C} \setminus X$;

(Q2) at each node X_k with $k \in \mathbb{N}$ and $k \geq n$, the first column of $Q(z)$ is analytic and the second column of $Q(z)$ has a simple pole with residue

$$\operatorname*{Res}_{z=X_k} Q(z) = \lim_{z \to X_k} Q(z) \begin{pmatrix} 0 & w(nz-\beta/2) \prod_{j=0}^{n-1} (z-X_j)^2 \\ 0 & 0 \end{pmatrix}; \tag{2.11}$$

at each node X_k with $k \in \mathbb{N}$ and $k < n$, the second column of $Q(z)$ is analytic and the first column of $Q(z)$ has a simple pole with residue

$$\operatorname*{Res}_{z=X_k} Q(z) = \lim_{z \to X_k} Q(z) \begin{pmatrix} 0 & 0 \\ \dfrac{1}{w(nz-\beta/2)} \prod_{\substack{j=0 \\ j \neq k}}^{n-1} (z-X_j)^{-2} & 0 \end{pmatrix}; \tag{2.12}$$

(Q3) for z bounded away from $\mathbb{X}$, $Q(z) = I + O(|z|^{-1})$ as $z \to \infty$.

Proof. On account of (2.10), (Q1) and (Q3) follow from (P1) and (P3), respectively.

Also from (2.10), we have

$$Q_{11}(z) = n^{-n} P_{11}(nz-\beta/2) \prod_{j=0}^{n-1} (z-X_j)^{-1}$$

and

$$Q_{12}(z) = n^{-n} P_{12}(nz-\beta/2) \prod_{j=0}^{n-1} (z-X_j).$$

At each node $z = X_k$ with $k \in \mathbb{N}$ and $k \geq n$, it is easily seen from (P2) that $Q_{11}(z)$ is analytic and $Q_{12}(z)$ has a simple pole, where the residue can be calculated as follows:

$$\begin{aligned} \operatorname*{Res}_{z=X_k} Q_{12}(z) &= n^{-n} \operatorname*{Res}_{z=X_k} P_{12}(nz-\beta/2) \prod_{j=0}^{n-1} (X_k - X_j) \\ &= n^{-n} w(nX_k - \beta/2) P_{11}(nX_k - \beta/2) \prod_{j=0}^{n-1} (X_k - X_j) \\ &= Q_{11}(X_k) w(nX_k - \beta/2) \prod_{j=0}^{n-1} (X_k - X_j)^2. \end{aligned}$$

Similarly, one can show from (P2) and (2.10) that $Q_{21}(z)$ is analytic and $Q_{22}(z)$ has a simple pole at X_k, $k \geq n$, with residue

$$\operatorname*{Res}_{z=X_k} Q_{22}(z) = Q_{21}(X_k) w(nX_k - \beta/2) \prod_{j=0}^{n-1} (X_k - X_j)^2.$$

This proves the first half of (Q2).

Now, we compute the singularities of $Q(z)$ at the nodes X_k with $k \in \mathbb{N}$ and $k < n$. First, it is easily seen from (P2) and (2.10) that $Q_{12}(z)$ can be analytically continued to the node $z = X_k$ and

$$\begin{aligned}
\lim_{z \to X_k} Q_{12}(z) &= \lim_{z \to X_k} n^{-n} P_{12}(nz - \beta/2) \prod_{j=0}^{n-1} (z - X_j) \\
&= n^{-n} \operatorname*{Res}_{z=X_k} P_{12}(nz - \beta/2) \prod_{\substack{j=0 \\ j \neq k}}^{n-1} (X_k - X_j) \\
&= n^{-n} w(nX_k - \beta/2) P_{11}(nX_k - \beta/2) \prod_{\substack{j=0 \\ j \neq k}}^{n-1} (X_k - X_j).
\end{aligned}$$

Furthermore, since $P_{11}(nz - \beta/2)$ is analytic at $z = X_k$ by (P2), the function $Q_{11}(z)$ has a simple pole at $z = X_k$ and from the last equation we obtain

$$\begin{aligned}
\operatorname*{Res}_{z=X_k} Q_{11}(z) &= n^{-n} P_{11}(nX_k - \beta/2) \prod_{\substack{j=0 \\ j \neq k}}^{n-1} (X_k - X_j)^{-1} \\
&= Q_{12}(X_k) w(nX_k - \beta/2)^{-1} \prod_{\substack{j=0 \\ j \neq k}}^{n-1} (X_k - X_j)^{-2}.
\end{aligned}$$

Similarly, we see from (P2) and (2.10) that $Q_{22}(z)$ is analytic and $Q_{21}(z)$ has a simple pole with residue

$$\begin{aligned}
\operatorname*{Res}_{z=X_k} Q_{21}(z) &= n^{-n} P_{21}(nX_k - \beta/2) \prod_{\substack{j=0 \\ j \neq k}}^{n-1} (X_k - X_j)^{-1} \\
&= Q_{22}(X_k) w(nX_k - \beta/2)^{-1} \prod_{\substack{j=0 \\ j \neq k}}^{n-1} (X_k - X_j)^{-2}.
\end{aligned}$$

This proves the second half of (Q2).

As in the proof of Proposition 2.1, the uniqueness again follows from Liouville's theorem. □

The purpose of the second transformation $Q \to R$ is to remove the poles in the interpolation problem for $Q(z)$. For any fixed $0 < c < 1$ and $1 \leq \beta < 2$, let $\delta_0 > 0$ be a small number that will be determined in Remark 2.9. Fix any $0 < \delta < \delta_0$, and define (cf. Figure 2.1)

$$R(z) := Q(z) \begin{pmatrix} 1 & 0 \\ a_{21}^{(\pm)} & 1 \end{pmatrix} \tag{2.13a}$$

for $\operatorname{Re} z \in (0,1)$ and $\operatorname{Im} z \in (0, \pm\delta)$, and

$$R(z) := Q(z) \begin{pmatrix} 1 & a_{12}^{(\pm)} \\ 0 & 1 \end{pmatrix} \tag{2.13b}$$

for $\operatorname{Re} z \in (0,1)$ and $\operatorname{Im} z \in (0, \pm\delta)$, and

$$R(z) := Q(z) \tag{2.13c}$$

for $\operatorname{Re} z \notin [0, \infty)$ or $\operatorname{Im} z \notin [-\delta, \delta]$, where

$$a_{12}^{(\pm)} := -\frac{n\pi w(nz - \beta/2) \prod_{j=0}^{n-1} (z - X_j)^2}{e^{\mp i\pi(nz-\beta/2)} \sin(n\pi z - \beta\pi/2)}, \tag{2.14}$$

and

$$a_{21}^{(\pm)} := -\frac{e^{\pm i\pi(nz-\beta/2)} \sin(n\pi z - \beta\pi/2)}{n\pi w(nz - \beta/2) \prod_{j=0}^{n-1} (z - X_j)^2}. \tag{2.15}$$

Lemma 2.3. *For each $k \in \mathbb{N}$, the singularity of $R(z)$ at the node $X_k = \frac{k+\beta/2}{n}$ is removable, that is,* $\operatorname*{Res}_{z=X_k} R(z) = 0$.

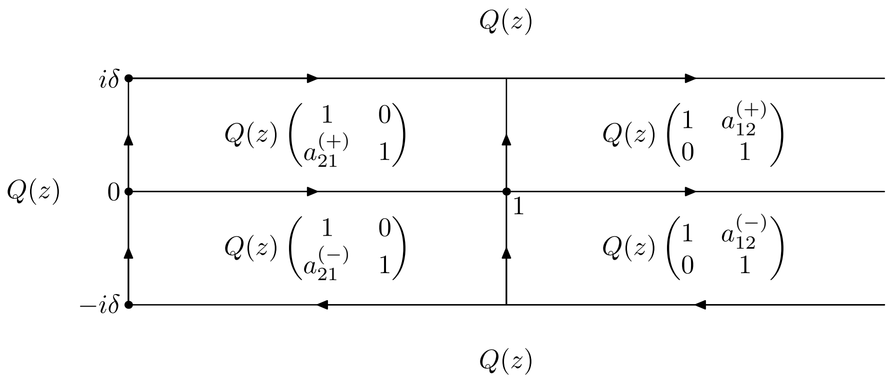

Figure 2.1 The transformation $Q \to R$ and the contour Σ_R.

Proof. For any $k \in \mathbb{N}$ with $k \geq n$, we have $X_k = \frac{k+\beta/2}{n} > 1$ since $1 \leq \beta < 2$. For any complex z with $\operatorname{Re} z \in (1, \infty)$ and $\operatorname{Im} z \in (0, \pm\delta)$, we obtain from (2.13) that $R_{11}(z) = Q_{11}(z)$ and

$$R_{12}(z) = Q_{12}(z) + Q_{11}(z)a_{12}^{(\pm)}. \tag{2.16}$$

The analyticity of the function $Q_{11}(z)$ at the node X_k is clear from (Q2) in Proposition 2.2. Hence, the function $R_{11}(z)$ is analytic. To show that the singularity of the function $R_{12}(z)$ at the node X_k is removable, we first note from (Q2) that

$$\operatorname*{Res}_{z=X_k} Q_{12}(z) = Q_{11}(X_k)w(nX_k - \beta/2)\prod_{j=0}^{n-1}(X_k - X_j)^2. \tag{2.17}$$

Furthermore, it follows from (2.14) that

$$\operatorname*{Res}_{z=X_k} a_{12}^{(\pm)} = -w(nX_k - \beta/2)\prod_{j=0}^{n-1}(X_k - X_j)^2. \tag{2.18}$$

Applying (2.17) and (2.18) to (2.16) yields $\operatorname*{Res}_{z=X_k} R_{12}(z) = 0$. Similarly, we can prove that the functions $R_{21}(z)$ and $R_{22}(z)$ are analytic at the node X_k.

Now, we consider the case $k \in \mathbb{N}$ with $k < n$. Since $1 \leq \beta < 2$, we have $X_k = \frac{k+\beta/2}{n} < 1$. For any z with $\operatorname{Re} z \in (0, 1)$ and $\operatorname{Im} z \in (0, \pm\delta)$, we obtain from (2.13) that $R_{12}(z) = Q_{12}(z)$ and

$$R_{11}(z) = Q_{11}(z) + Q_{12}(z)a_{21}^{(\pm)}. \tag{2.19}$$

From (Q2) in Proposition 2.2 we see that the function $Q_{12}(z)$, and hence the function $R_{12}(z)$, is analytic at the node X_k. Moreover, we have

$$\operatorname*{Res}_{z=X_k} Q_{11}(z) = Q_{12}(X_k)\frac{1}{w(nX_k-\beta/2)}\prod_{\substack{j=0\\ j\neq k}}^{n-1}(X_k-X_j)^{-2}. \tag{2.20}$$

Since

$$\operatorname*{Res}_{z=X_k} a_{21}^{(\pm)} = \frac{-1}{w(nX_k-\beta/2)}\prod_{\substack{j=0\\ j\neq k}}^{n-1}(X_k-X_j)^{-2}$$

by (2.15), we obtain from (2.19) and (2.20) that $\operatorname*{Res}_{z=X_k} R_{11}(z)=0$. The analyticity of the second row in $R(z)$ at the node X_k can be verified similarly. This ends our proof. □

From the definition of $R(z)$ in (2.13) and the analyticity condition (Q1) of $Q(z)$ in Proposition 2.2, it is easily seen that $R(z)$ is analytic in $\mathbb{C}\setminus\Sigma_R$, where Σ_R is the oriented contour shown in Figure 2.1. Denote by $R_+(z)$ the limiting value taken by $R(z)$ on Σ_R from the left and by $R_-(z)$ taken from the right. We intend to calculate the jump matrix $J_R(z) := R_-(z)^{-1}R_+(z)$ on the contour Σ_R. For convenience, we introduce the two functions

$$v(z) := -z\log c \tag{2.21}$$

and

$$W(z) := 2in\pi w(nz-\beta/2)e^{nv(z)} = \frac{2in\pi\Gamma(nz+\beta/2)c^{-\beta/2}}{\Gamma(nz+1-\beta/2)}. \tag{2.22}$$

Consequently, the functions $a_{12}^{\pm}$ and $a_{21}^{\pm}$ defined in (2.14) and (2.15) become

$$a_{12}^{(\pm)} = -\frac{W(z)e^{-nv(z)}\prod_{j=0}^{n-1}(z-X_j)^2}{2i\sin(n\pi z-\beta\pi/2)e^{\mp i\pi(nz-\beta/2)}}, \tag{2.23}$$

and

$$a_{21}^{(\pm)} = -\frac{2i\sin(n\pi z-\beta\pi/2)e^{\pm i\pi(nz-\beta/2)}}{W(z)e^{-nv(z)}\prod_{j=0}^{n-1}(z-X_j)^2}. \tag{2.24}$$

It is easily seen from (2.23) and (2.24) that

$$a_{12}^{(\pm)} \cdot a_{21}^{(\pm)} = e^{\pm 2i\pi(nz-\beta/2)}, \tag{2.25}$$

and

$$a_{21}^{(+)} - a_{21}^{(-)} = \frac{4\sin^2(n\pi z - \beta\pi/2)}{W(z)e^{-nv(z)}\prod_{j=0}^{n-1}(z-X_j)^2}, \tag{2.26}$$

and

$$a_{12}^{(+)} - a_{12}^{(-)} = -W(z)e^{-nv(z)}\prod_{j=0}^{n-1}(z-X_j)^2. \tag{2.27}$$

The jump conditions of $R(z)$ is given below.

Proposition 2.4. *On the contour Σ_R, the jump matrix $J_R(z) := R_-(z)^{-1}R_+(z)$ has the following explicit expressions. For $z = 1 + i\operatorname{Im} z$ with $\operatorname{Im} z \in (0, \pm\delta)$, we have*

$$J_R(z) = \begin{pmatrix} 1 - e^{\pm 2i\pi(nz-\beta/2)} & -a_{12}^{(\pm)} \\ a_{21}^{(\pm)} & 1 \end{pmatrix}. \tag{2.28}$$

On the positive real line, we have

$$J_R(x) = \begin{pmatrix} 1 & 0 \\ a_{21}^{(+)} - a_{21}^{(-)} & 1 \end{pmatrix} \tag{2.29a}$$

for $x \in (0,1)$, and

$$J_R(x) = \begin{pmatrix} 1 & a_{12}^{(+)} - a_{12}^{(-)} \\ 0 & 1 \end{pmatrix} \tag{2.29b}$$

for $x \in (1,\infty)$. Furthermore, we have

$$J_R(z) = \begin{pmatrix} 1 & 0 \\ -a_{21}^{(\pm)} & 1 \end{pmatrix} \tag{2.30a}$$

for $z \in (0, \pm i\delta) \cup (\pm i\delta, 1 \pm i\delta)$, and

$$J_R(z) = \begin{pmatrix} 1 & -a_{12}^{(\pm)} \\ 0 & 1 \end{pmatrix} \tag{2.30b}$$

for $z = \operatorname{Re} z \pm i\delta$ with $\operatorname{Re} z \in (1,\infty)$.

Proof. For $z = 1 + i\operatorname{Im} z$ with $\operatorname{Im} z \in (0, \delta)$, we obtain from (2.13) that

$$R_+(z) = Q(z)\begin{pmatrix} 1 & 0 \\ a_{21}^{(+)} & 1 \end{pmatrix},$$

and

$$R_-(z) = Q(z)\begin{pmatrix} 1 & a_{12}^{(+)} \\ 0 & 1 \end{pmatrix}.$$

Thus, we have from (2.25)

$$J_R(z) = \begin{pmatrix} 1 & -a_{12}^{(+)} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ a_{21}^{(+)} & 1 \end{pmatrix} = \begin{pmatrix} 1 - e^{2i\pi(nz-\beta/2)} & -a_{12}^{(+)} \\ a_{21}^{(+)} & 1 \end{pmatrix}.$$

Similarly, for $z = 1 + i\operatorname{Im} z$ with $\operatorname{Im} z \in (-\delta, 0)$, we obtain from (2.13) and (2.25) that

$$J_R(z) = \begin{pmatrix} 1 & -a_{12}^{(-)} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ a_{21}^{(-)} & 1 \end{pmatrix} = \begin{pmatrix} 1 - e^{-2i\pi(nz-\beta/2)} & -a_{12}^{(-)} \\ a_{21}^{(-)} & 1 \end{pmatrix}.$$

Hence, formula (2.28) is proved.

For any $x > 0$ with $x \notin \mathbb{X}$, we obtain from (2.13) and (2.22) that

$$R_\pm(x) = Q(x)\begin{pmatrix} 1 & 0 \\ a_{21}^{(\pm)} & 1 \end{pmatrix}$$

for $x \in (0, 1)$, and

$$R_\pm(x) = Q(x)\begin{pmatrix} 1 & a_{12}^{(\pm)} \\ 0 & 1 \end{pmatrix}$$

for $x \in (1, \infty)$. A simple calculation gives (2.29). Since $R(z)$ has no singularity at $\mathbb{X}$ (Lemma 2.3), formula (2.29) remains valid when $x \in \mathbb{X}$.

Finally, (2.30) is clear from (2.13). This completes our proof. $\square$

Proposition 2.5. *The matrix-valued function $R(z)$ defined in (2.13) is the unique solution to the following Riemann-Hilbert problem:*

(R1) $R(z)$ *is analytic in* $\mathbb{C} \setminus \Sigma_R$*;*

(R2) for $z \in \Sigma_R$, $R_+(z) = R_-(z)J_R(z)$, *where the jump matrix* $J_R(z)$ *is given in Proposition 2.4;*

(R3) for $z \in \mathbb{C} \setminus \Sigma_R$, $R(z) = I + O(|z|^{-1})$ *as* $z \to \infty$*.*

Proof. Condition (R1) follows from the analyticity condition (Q1) in Proposition 2.2 and the definition of $R(z)$ in (2.13). Proposition 2.4 gives (R2). Furthermore, the normalization condition (Q3) in Proposition 2.2 yields (R3). The uniqueness of solution is again a direct consequence of Liouville's theorem. □

For the preparation of the third transformation $R \to S$, we investigate the equilibrium measure corresponding to the Meixner polynomials. In the existing literature, the equilibrium measure is usually obtained by solving a minimization problem of a certain quadratic functional (cf. [2, 3, 6, 7]). Here, we prefer to use the method introduced by Kuijlaars and Van Assche [20].

Consider the monic polynomials $q_{n,N}(x) := N^{-n}\pi_n(Nx-\beta/2)$, where $N \in \mathbb{N}$. From (2.2), we have

$$xq_{n,N}(x) = q_{n+1,N}(x) + \frac{(n+\beta/2)(1+c)}{N(1-c)}q_{n,N}(x) + \frac{n(n+\beta-1)c}{N^2(1-c)^2}q_{n,N}(x).$$

The coefficients $\frac{(n+\beta/2)(1+c)}{N(1-c)}$ and $\frac{n(n+\beta-1)c}{N^2(1-c)^2}$ correspond to the recurrence coefficients $b_{n,N}$ and $a_{n,N}^2$ in [20, (1.6)]. Suppose $n/N \to t > 0$ as $n \to \infty$. It can be shown that

$$\frac{(n+\beta/2)(1+c)}{N(1-c)} \to \frac{1+c}{1-c}t, \qquad \sqrt{\frac{n(n+\beta-1)c}{N^2(1-c)^2}} \to \frac{\sqrt{c}}{1-c}t.$$

Define two constants

$$a := \frac{1-\sqrt{c}}{1+\sqrt{c}} \qquad \text{and} \qquad b := \frac{1+\sqrt{c}}{1-\sqrt{c}}, \tag{2.31}$$

and note that $ab = 1$. The functions $\alpha(t)$ and $\beta(t)$ in [20, (1.8)] are equal to at and bt respectively. Therefore, from Theorem 1.4 in [20], the asymptotic zero distribution of $q_{n,N}(x)$ with $n/N \to t > 0$ is given by

$$\mu_t(x) = \frac{1}{t}\int_0^t \omega_{[as,bs]}(x)ds,$$

where

$$\frac{d\omega_{[as,bs]}(x)}{dx} = \frac{1}{\pi\sqrt{(bs-x)(x-as)}}$$

for $x \in (as, bs)$, and $\frac{d\omega_{[as,bs]}(x)}{dx} = 0$ elsewhere; see [20, (1.4)]. Thus, the density function of $\mu_t(x)$ is

$$\frac{d\mu_t(x)}{dx} = \frac{1}{\pi t}\int_{ax}^{bx} \frac{ds}{\sqrt{(bs-x)(x-as)}}$$

for $x \in [0, at]$, and

$$\frac{d\mu_t(x)}{dx} = \frac{1}{\pi t}\int_{ax}^{t} \frac{ds}{\sqrt{(bs-x)(x-as)}}$$

for $x \in [at, bt]$. We only need to consider the special case $N = n$. Therefore, when $t = 1$, the density function becomes

$$\rho(x) := \frac{d\mu_1(x)}{dx} = \begin{cases} 1, & 0 < x < a, \\ \\ \dfrac{1}{\pi}\arccos\dfrac{x(b+a)-2}{x(b-a)}, & a < x < b, \end{cases} \tag{2.32}$$

where we have used the equality

$$\int_{ax}^{1} \frac{ds}{\sqrt{(bs-x)(x-as)}} = \arccos\frac{x(b+a)-2}{x(b-a)};$$

see (4.1) in Appendix. The *equilibrium measure* for our problem is $d\mu_1(x) = \rho(x)dx$. Note that the constants a and b defined in (2.31) are the same as the constants α_- and α_+ in [16, (2.6)]. They are called the *Mhaskar-Rakhmanov-Saff numbers* or the *turning points*. We now define the so-called g *– function.*

$$g(z) := \int_0^b \log(z-x)\rho(x)dx \tag{2.33}$$

for $z \in \mathbb{C} \setminus (-\infty, b]$. On account of (2.31) and (2.32), the derivative of $g(z)$ can be calculated as shown below (cf. (4.23) in Appendix).

$$\begin{aligned} g'(z) &= \int_0^b \frac{1}{z-x}\rho(x)dx \\ &= -\log \frac{z(b+a) - 2 + 2\sqrt{(z-a)(z-b)}}{z(b-a)} + \frac{-\log c}{2}. \end{aligned} \tag{2.34}$$

Proposition 2.6. *The function $g'(z)$ given in (2.34) is the unique solution to the following scalar Riemann-Hilbert problem:*

(g1) $g'(z)$ is analytic in $\mathbb{C} \setminus [0, b]$;

(g2) denoting the limiting value taken by $g'(z)$ on the real line from the upper half plane by $g'_+(x)$ and that taken from the lower half plane by $g'_-(x)$, the function $g'(z)$ satisfies the jump conditions:

$$g'_+(x) - g'_-(x) = -2\pi i, \qquad 0 < x < a, \tag{2.35}$$

$$g'_+(x) + g'_-(x) = -\log c, \qquad a < x < b; \tag{2.36}$$

(g3) $g'(z) = \dfrac{1}{z} + O(|z|^{-2})$, as $z \to \infty$.

Proof. The analyticity condition (g1) is trivial by (2.34). The normalization condition (g3) follows from the fact

$$\int_0^b \rho(x)dx = 1;$$

see (4.22) in Appendix. For $0 < x < a$, we obtain from (2.34) that

$$g'_\pm(x) = -\log \frac{-x(b+a) + 2 + 2\sqrt{(a-x)(b-x)}}{x(b-a)} \mp i\pi + \frac{-\log c}{2}.$$

Therefore, the relation (2.35) follows. For $a < x < b$, we obtain from (2.34) that

$$g'_\pm(x) = -\log \frac{x(b+a) - 2 \pm 2i\sqrt{(x-a)(b-x)}}{x(b-a)} + \frac{-\log c}{2}.$$

Therefore, the relation (2.36) follows. Finally, the uniqueness is again guaranteed by Liouville's theorem. □

Remark 2.7. *From (2.32) we observe that the equilibrium measure of the Meixner polynomials corresponds to the saturated-band-void configuration defined in [3]; see also [6]. We point out that the equilibrium measure $\rho(x)dx$ can be solved in a different way, that is, regard $\rho(x)dx$ as the measure which satisfies the constraint*

$$0 \le \rho(x) \le 1$$

on the interval $[0,\infty)$, and minimizes the quadratic functional

$$\int_0^\infty \int_0^\infty \log \frac{1}{|x-y|}\rho(x)\rho(y)dxdy + \int_0^\infty v(x)\rho(x)dx,$$

where $v(x)$ is defined in (2.21); see [9, 24]. Following the procedure in [3, Section B.3], we first show that the Mhaskar-Rakhmanov-Saff numbers a and b are the solutions to the following equations

$$\int_a^b \frac{v'(x)}{\sqrt{(x-a)(b-x)}}dx - \int_0^a \frac{2\pi}{\sqrt{(a-x)(b-x)}}dx = 0,$$
$$\int_a^b \frac{xv'(x)}{\sqrt{(x-a)(b-x)}}dx - \int_0^a \frac{2\pi x}{\sqrt{(a-x)(b-x)}}dx = 2\pi.$$

In the second step we find that the function $g'(z)$, which corresponds to the function $F(z)$ in [3, (710)], has the explicit expression

$$g'(z) = \int_0^a \frac{\sqrt{(z-a)(z-b)}}{\sqrt{(a-x)(b-x)}}\frac{dx}{x-z} - \int_a^b \frac{\sqrt{(z-a)(z-b)}}{\sqrt{(x-a)(b-x)}}\frac{v'(x)dx}{2\pi(x-z)}.$$

Finally, the equilibrium measure $\rho(x)dx$ is supported on the interval $[0,b]$ and

$$\rho(x) = \frac{g'_-(x) - g'_+(x)}{2\pi i}$$

for $x \in [0,b]$. A direct calculation shows that $\rho(x) = 1$ on the saturated interval $[0,a]$, and $\rho(x) = \frac{1}{\pi}\arccos\frac{x(b+a)-2}{x(b-a)}$ on the band $[a,b]$. This agrees with formula (2.32).

Recall that $v(z) = -z\log c$ in (2.21). It is easily seen from (2.34) that

$$-g'(\zeta) + \frac{v'(\zeta)}{2} = \log\frac{\zeta(b+a) - 2 + 2\sqrt{(\zeta-a)(\zeta-b)}}{\zeta(b-a)}$$

for $\zeta \in \mathbb{C} \setminus (-\infty, b]$. We introduce the so-called ϕ – *function.*

$$\begin{aligned}\phi(z) &:= \int_b^z (-g'(\zeta) + \frac{v'(\zeta)}{2})d\zeta \\ &= \int_b^z \log \frac{\zeta(b+a) - 2 + 2\sqrt{(\zeta-a)(\zeta-b)}}{\zeta(b-a)} d\zeta \end{aligned} \tag{2.37}$$

for $z \in \mathbb{C} \setminus (-\infty, b]$. From the definition we observe

$$\begin{aligned}\phi(z) &= -g(z) + v(z)/2 + g(b) - v(b)/2 \\ &= -g(z) + v(z)/2 + l/2,\end{aligned}$$

where

$$l := 2g(b) - v(b) = 2\log\frac{b-a}{4} - 2 \tag{2.38}$$

is called the *Lagrange multiplier.* The calculation of the last equality is given in Appendix; see (4.24). We also introduce the so-called $\widetilde{\phi}$ – *function.*

$$\begin{aligned}\widetilde{\phi}(z) &:= \int_a^z \log \frac{-\zeta(b+a) + 2 - 2\sqrt{(\zeta-a)(\zeta-b)}}{\zeta(b-a)} d\zeta \\ &= \phi(z) \pm i\pi(1-z)\end{aligned} \tag{2.39}$$

for $z \in \mathbb{C}_{\pm}$. Note that the integrand of the integral in the last equation can be analytically continued to the interval $(0, a)$. Thus the function $\widetilde{\phi}(z)$ is analytic in $(0, a)$; see also (2.42) below. We now provide some important properties of the g –, ϕ – and $\widetilde{\phi}$ – functions.

Proposition 2.8. *Let the functions g, ϕ and $\widetilde{\phi}$ be defined as in (2.33), (2.37) and (2.39), respectively. Recall from (2.21) and (2.38) that $v(z) = -z\log c$ and $l = 2\log\frac{b-a}{4} - 2$. We have*

$$2g(z) + 2\phi(z) - v(z) - l = 0 \tag{2.40}$$

for all $z \in \mathbb{C} \setminus (-\infty, b]$. Denote the boundary value taken by $\phi(z)$ on the real line from the upper half plane by ϕ_+ and that taken from the lower half plane by ϕ_-.

We have

$$\phi_+ = \begin{cases} \phi_- - 2i\pi(1-x) & : \quad 0 < x < a, \\ -\phi_- & : \quad a < x < b, \\ \phi_- & : \quad x > b. \end{cases} \tag{2.41}$$

Denote the boundary value taken by $\widetilde{\phi}(z)$ *on the real line from the upper half plane by* $\widetilde{\phi}_+$ *and that taken from the lower half plane by* $\widetilde{\phi}_-$*. We have*

$$\widetilde{\phi}_+ = \begin{cases} \widetilde{\phi}_- & : \quad 0 < x < a, \\ -\widetilde{\phi}_- & : \quad a < x < b, \\ \widetilde{\phi}_- + 2i\pi(1-x) & : \quad x > b. \end{cases} \tag{2.42}$$

Denote the boundary value taken by $g(z)$ *on the real line from the upper half plane by* g_+ *and that taken from the lower half plane by* g_-*. We have*

$$g_+ + g_- - v - l = \begin{cases} -2\phi_+ - 2i\pi(1-x) & : \quad 0 < x < a, \\ 0 & : \quad a < x < b, \\ -2\phi & : \quad x > b. \end{cases} \tag{2.43}$$

Furthermore, we have

$$g_+ - g_- = \begin{cases} 2i\pi(1-x) & : \quad 0 < x < a, \\ -2\phi_+ = 2\phi_- & : \quad a < x < b, \\ 0 & : \quad x > b. \end{cases} \tag{2.44}$$

For any small $\varepsilon > 0$ *and* $z \in U(b, \varepsilon) := \{z \in \mathbb{C} : |z - b| < \varepsilon\}$*, we have*

$$\phi(z) = \frac{4(z-b)^{3/2}}{3b\sqrt{b-a}} + O(\varepsilon^2). \tag{2.45}$$

For any small $\varepsilon > 0$ *and* $z \in U(a, \varepsilon) := \{z \in \mathbb{C} : |z - a| < \varepsilon\}$*, we have*

$$\widetilde{\phi}(z) = \frac{-4(a-z)^{3/2}}{3a\sqrt{b-a}} + O(\varepsilon^2). \tag{2.46}$$

For any small $\varepsilon > 0$ *and* $x > b + \varepsilon$*, we have*

$$\phi(x) > \phi(b+\varepsilon) = \frac{4\varepsilon^{3/2}}{3b\sqrt{b-a}} + O(\varepsilon^2). \tag{2.47}$$

For any small $\varepsilon > 0$ *and* $0 < x < a - \varepsilon$*, we have*

$$\widetilde{\phi}(x) < \widetilde{\phi}(a-\varepsilon) = \frac{-4\varepsilon^{3/2}}{3a\sqrt{b-a}} + O(\varepsilon^2). \tag{2.48}$$

For any $x \in (a,b)$ and sufficiently small $y > 0$, we have

$$\operatorname{Re}\phi(x \pm iy) = -y \arccos \frac{x(b+a)-2}{x(b-a)} + O(y^2), \tag{2.49}$$

$$\operatorname{Re}\widetilde{\phi}(x \pm iy) = y \arccos \frac{2-x(b+a)}{x(b-a)} + O(y^2). \tag{2.50}$$

For any $x \in (b,\infty)$ and sufficiently small $y > 0$, we have

$$\operatorname{Re}\phi(x \pm iy) = \phi(x) + O(y^2), \qquad \operatorname{Re}\widetilde{\phi}(x \pm iy) = \phi(x) + \pi y + O(y^2). \tag{2.51}$$

For any $x \in (0,a)$ and sufficiently small $y > 0$, we have

$$\operatorname{Re}\widetilde{\phi}(x \pm iy) = \widetilde{\phi}(x) + O(y^2), \qquad \operatorname{Re}\phi(x \pm iy) = \widetilde{\phi}(x) - \pi y + O(y^2). \tag{2.52}$$

Proof. The relation (2.40) follows from the definition of ϕ – function in (2.37) and Lagrange multiplier in (2.38).

To prove (2.41), we first see from (2.37) that $\phi(z)$ is analytic for $z \in \mathbb{C} \setminus (-\infty, b]$. Thus, we have $\phi_+(x) - \phi_-(x) = 0$ for $x > b$. Moreover, we obtain from (2.37) that for $a < x < b$,

$$\phi_\pm(x) = \int_b^x \log \frac{s(b+a) - 2 \pm 2i\sqrt{(s-a)(b-s)}}{s(b-a)} ds,$$

which implies $\phi_+(x) + \phi_-(x) = 0$. On the other hand, for $0 < x < a$, it follows from (2.37) that

$$\begin{aligned}\phi_\pm(x) =& \int_b^a \log \frac{s(b+a) - 2 \pm 2i\sqrt{(s-a)(b-s)}}{s(b-a)} ds \\ &+ \int_a^x (\log \frac{-s(b+a) + 2 + 2i\sqrt{(a-s)(b-s)}}{s(b-a)} \pm i\pi) ds.\end{aligned}$$

In view of the equality (cf. (4.4) in Appendix)

$$\begin{aligned}\int_b^a \log \frac{s(b+a) - 2 \pm 2i\sqrt{(s-a)(b-s)}}{s(b-a)} ds =& \mp i \int_a^b \arccos \frac{s(b+a)-2}{s(b-a)} ds \\ =& \mp i\pi(1-a),\end{aligned}$$

we have

$$\phi_+(x) - \phi_-(x) = -2i\pi(1-a) + 2i\pi(x-a) = -2i\pi(1-x)$$

for $0 < x < a$. This ends the proof of (2.41).

Applying (2.39) to (2.41) gives (2.42).

From (2.40) we have

$$g_+(x) + g_-(x) - v(x) - l = -\phi_+(x) - \phi_-(x)$$

for $x \in \mathbb{R}$. Hence, the relation (2.43) follows immediately from (2.41).

It is easily seen from (2.33) that the function $g(z)$ is analytic for $z \in \mathbb{C} \setminus (-\infty, b]$. Coupling (2.40) and (2.41) yields

$$g_+ - g_- = \phi_- - \phi_+ = -2\phi_+ = 2\phi_-$$

for $a < x < b$. On the other hand, a combination of (2.37), (2.40) and (2.41) gives

$$\begin{aligned} g_+(a) - g_-(a) &= \phi_-(a) - \phi_+(a) \\ &= 2\phi_-(a) \\ &= 2\int_b^a \log \frac{s(b+a) - 2 - 2i\sqrt{(s-a)(b-s)}}{s(b-a)} ds \\ &= 2i\int_a^b \arccos \frac{s(b+a)-2}{s(b-a)} ds = 2i\pi(1-a). \end{aligned}$$

Coupling this with (2.35) gives

$$g_+(x) - g_-(x) = g_+(a) - g_-(a) + 2i\pi(a-x) = 2i\pi(1-x)$$

for $0 < x < a$. This completes the proof of (2.44).

For any small $\varepsilon > 0$ and $z \in U(b,\varepsilon) := \{z \in \mathbb{C} : |z-b| < \varepsilon\}$, from (2.37) we have

$$\begin{aligned} \phi(z) &= \int_b^z \log\left(1 + \frac{2\sqrt{(b-a)(\zeta-b)}}{b(b-a)} + O(\varepsilon)\right) d\zeta \\ &= \frac{4(z-b)^{3/2}}{3b\sqrt{b-a}} + O(\varepsilon^2). \end{aligned}$$

Here again, we have used the fact that $ab = 1$. This gives (2.45).

For any small $\varepsilon > 0$ and $z \in U(a, \varepsilon) := \{z \in \mathbb{C} : |z - a| < \varepsilon\}$, from (2.39) and the fact $ab = 1$ we have

$$\begin{aligned}\phi(z) &= \int_a^z \log\left(1 + \frac{2\sqrt{(a-\zeta)(b-a)}}{a(b-a)} + O(\varepsilon)\right) d\zeta \\ &= \frac{-4(a-z)^{3/2}}{3a\sqrt{b-a}} + O(\varepsilon^2).\end{aligned}$$

This gives (2.46).

From (2.37) and (2.39), we have

$$\phi'(x) = \log\frac{x(b+a) - 2 + 2\sqrt{(x-a)(x-b)}}{x(b-a)} > 0$$

for $x > b$ and

$$\widetilde{\phi}'(x) = \log\frac{-\zeta(b+a) + 2 - 2\sqrt{(\zeta-a)(\zeta-b)}}{\zeta(b-a)} > 0$$

for $0 < x < a$. Consequently, $\phi(x) > \phi(b+\varepsilon)$ for $x > b + \varepsilon$, and $\widetilde{\phi}(x) < \widetilde{\phi}(a-\varepsilon)$ for $0 < x < a - \varepsilon$. Therefore, the formulas (2.47) and (2.48) follow from (2.45) and (2.46), respectively.

It is easily seen from (2.37) and (2.39) that $\phi_\pm(x)$ and $\widetilde{\phi}_\pm(x)$ are purely imaginary for $a < x < b$. Hence, for any $x \in (a, b)$ and sufficiently small $y > 0$, we have

$$\begin{aligned}\operatorname{Re}\phi(x \pm iy) &= \operatorname{Re}\int_x^{x\pm iy} \log\frac{\zeta(b+a) - 2 + 2\sqrt{(\zeta-a)(\zeta-b)}}{\zeta(b-a)} d\zeta \\ &= \operatorname{Re}\int_x^{x\pm iy}\left(\pm i \arccos\frac{x(b+a)-2}{x(b-a)} + O(y)\right) d\zeta \\ &= -y \arccos\frac{x(b+a)-2}{x(b-a)} + O(y^2),\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Re}\widetilde{\phi}(x \pm iy) &= \operatorname{Re}\int_x^{x\pm iy} \log \frac{-\zeta(b+a)+2-2\sqrt{(\zeta-a)(\zeta-b)}}{\zeta(b-a)} d\zeta \\
&= \operatorname{Re}\int_x^{x\pm iy} \left(\mp i \arccos \frac{2-x(b+a)}{x(b-a)} + O(y) \right) d\zeta \\
&= y \arccos \frac{2-x(b+a)}{x(b-a)} + O(y^2).
\end{aligned}$$

This ends the proof of (2.49) and (2.50).

For any $x \in (b, \infty)$ and sufficiently small $y > 0$, from (2.37) we have

$$\phi(x \pm iy) - \phi(x) = \int_x^{x\pm iy} \log \frac{\zeta(b+a)-2+2\sqrt{(\zeta-a)(\zeta-b)}}{\zeta(b-a)} d\zeta.$$

Since the integral on the right-hand side equals to

$$\pm iy \log \frac{x(b+a)-2+2\sqrt{(x-a)(x-b)}}{x(b-a)} + O(y^2),$$

it follows that

$$\operatorname{Re}\phi(x \pm iy) = \phi(x) + O(y^2).$$

Moreover, we obtain from (2.39) and the last equation that

$$\operatorname{Re}\widetilde{\phi}(x \pm iy) = \operatorname{Re}\phi(x \pm iy) + \pi y = \phi(x) + \pi y + O(y^2),$$

thus proving (2.51).

Similarly, for any $x \in (0, a)$ and sufficiently small $y > 0$, we have from (2.39)

$$\widetilde{\phi}(x \pm iy) - \widetilde{\phi}(x) = \int_x^{x\pm iy} \log \frac{-\zeta(b+a)+2-2\sqrt{(\zeta-a)(\zeta-b)}}{\zeta(b-a)} d\zeta.$$

Since the integral on the right-hand side equals to

$$\pm iy \log \frac{-x(b+a)+2+2\sqrt{(a-x)(b-x)}}{x(b-a)} + O(y^2),$$

it follows that

$$\operatorname{Re}\widetilde{\phi}(x \pm iy) = \widetilde{\phi}(x) + O(y^2).$$

Moreover, we obtain from (2.39) and the last equation that

$$\operatorname{Re}\phi(x \pm iy) = \operatorname{Re}\widetilde{\phi}(x \pm iy) - \pi y = \widetilde{\phi}(x) - \pi y + O(y^2),$$

thus proving (2.52). □

Remark 2.9. *Recall that the constant $\delta_0 > 0$ introduced in the definition of $R(z)$ has not been determined; see (2.13). Fix any $0 < c < 1$ and $1 \le \beta < 2$, we choose $\delta_0 > 0$ to be sufficiently small such that the function $\phi(z)^{2/3}$ is analytic in the open disk $U(b, \delta_0)$ and the function $\widetilde{\phi}(z)^{2/3}$ is analytic in the open disk $U(a, \delta_0)$. We also require δ_0 to be so small that the formulas (2.45)-(2.52) in Proposition 2.8 are valid whenever $\varepsilon, y \in (0, \delta_0)$. The existence of such a positive constant δ_0 is obvious. Furthermore, since the functions $\phi(z)$ and $\widetilde{\phi}(z)$ depend only on the constants c and β. the constant δ_0 is independent of the polynomial degree n.*

For the sake of simplicity, we introduce some auxiliary functions. Define

$$E(z) := \left(\frac{z-1}{z}\right)^{\frac{1-\beta}{2}} \exp\left\{-n\int_0^1 \log(z-x)dx\right\} \prod_{k=0}^{n-1}(z - X_k) \tag{2.53}$$

for $z \in \mathbb{C} \setminus [0, 1]$, and

$$\widetilde{E}(z) := \frac{\pm\, iE(z)e^{\mp i\pi(nz-\beta/2)}}{2\sin(n\pi z - \beta\pi/2)} \tag{2.54}$$

for $z \in \mathbb{C}_\pm$, and

$$H(z) := \left(\frac{z}{z-1}\right)^{1-\beta} W(z) \tag{2.55}$$

for $z \in \mathbb{C} \setminus [0, 1]$, and

$$\widetilde{H}(z) := \left(\frac{z}{1-z}\right)^{1-\beta} W(z) \tag{2.56}$$

for $z \in \mathbb{C} \setminus (-\infty, 0] \cup [1, \infty)$, where $W(z)$ is defined in (2.22). We also recall from (2.9) that $X_k = \frac{k+\beta/2}{n}$. The properties of the above auxiliary functions are given in the following lemma.

Lemma 2.10. *The function* $\widetilde{E}(z)$ *defined in (2.54) can be analytically continued to the interval* $(0,1)$*. Moreover, for any* $0<x<1$*, we have*

$$\widetilde{E}(x)^2 = \frac{E_+(x)E_-(x)}{4\sin^2(n\pi x-\beta\pi/2)}. \tag{2.57}$$

For any $z\in\mathbb{C}_\pm$*, we have*

$$E(z)/\widetilde{E}(z) = \mp 2ie^{\pm i\pi(nz-\beta/2)}\sin(n\pi z-\beta\pi/2) = 1-e^{\pm 2i\pi(nz-\beta/2)}, \tag{2.58}$$

$$\widetilde{H}(z) = H(z)e^{\pm i\pi(1-\beta)} = -H(z)e^{\mp i\pi\beta}. \tag{2.59}$$

As $n\to\infty$*, we have* $E(z)\sim 1$ *uniformly for* z *bounded away from the interval* $[0,1]$ *and* $E(z)/\widetilde{E}(z)\sim 1$ *uniformly for* z *bounded away from the real line.*

Proof. For $0<x<1$, from (2.53) we have

$$E_\pm(x) = \left(\frac{1-x}{x}\right)^{\frac{1-\beta}{2}} e^{\pm i\pi(1-\beta)/2}\exp\left\{-n\int_0^1 \log|x-s|ds\right\} e^{\mp n\pi i(1-x)}\prod_{k=0}^{n-1}(z-X_k).$$

Consequently, we obtain $E_+(x)/E_-(x) = -e^{2i\pi(nx-\beta/2)}$. Therefore, it is readily seen from (2.54) that $\widetilde{E}_+(x)=\widetilde{E}_-(x)$ on the interval $(0,1)$. Moreover, we have

$$\widetilde{E}^2(x) = \frac{E_+(x)E_-(x)}{4\sin^2(n\pi x-\beta\pi/2)}, \qquad 0<x<1.$$

This gives (2.57).

The relation (2.58) follows from (2.54). The relation (2.59) follows from (2.55) and (2.56).

Let z be bounded away from the interval $[0,1]$. Since $X_k=\frac{k+\beta/2}{n}$ by (2.9), we have

$$\begin{aligned}\prod_{k=0}^{n-1}(z-X_k) &= \prod_{k=0}^{n-1}\left(z-\frac{k+\beta/2}{n}\right)\\ &= \frac{\Gamma(nz-\beta/2+1)}{n^n\Gamma(nz-\beta/2-n+1)}.\end{aligned}$$

Using Stirling's formula, it follows that

$$\begin{aligned}
\prod_{k=0}^{n-1}(z-X_k) &\sim \frac{\sqrt{2\pi(nz-\beta/2)}(\frac{nz-\beta/2}{e})^{nz-\beta/2}}{n^n\sqrt{2\pi(nz-\beta/2-n)}(\frac{nz-\beta/2-n}{e})^{nz-\beta/2-n}} \\
&= \frac{(nz-\beta/2)^{\frac{1-\beta}{2}}(\frac{nz-\beta/2}{nz})^{nz}(nz)^{nz}}{n^n(nz-\beta/2-n)^{\frac{1-\beta}{2}}(\frac{nz-\beta/2-n}{nz-n})^{nz-n}(nz-n)^{nz-n}e^n} \\
&\sim \left(\frac{z}{z-1}\right)^{\frac{1-\beta}{2}}\left(\frac{z}{z-1}\right)^{nz}\left(\frac{z-1}{e}\right)^n
\end{aligned}$$

as $n\to\infty$. In view of the equality

$$\exp\left\{-n\int_0^1 \log(z-x)dx\right\} = \frac{e^n(z-1)^{nz}}{z^{nz}(z-1)^n},$$

we then obtain from (2.53) that

$$E(z)\sim\left(\frac{z-1}{z}\right)^{\frac{1-\beta}{2}}\frac{e^n(z-1)^{nz}}{z^{nz}(z-1)^n}\left(\frac{z}{z-1}\right)^{\frac{1-\beta}{2}}\left(\frac{z}{z-1}\right)^{nz}\left(\frac{z-1}{e}\right)^n=1$$

as $n\to\infty$.

Finally, as $n\to\infty$, it is easily seen from (2.58) that $E(z)/\widetilde{E}(z)\sim 1$ uniformly for z bounded away from the real line. This ends the proof of the lemma. □

Recalling the definition of $g(z)$ in (2.33), we introduce the function

$$\begin{aligned}
G(z) &:= ng(z)-n\int_0^1\log(z-x)dx \\
&= n\int_0^b \log(z-x)\rho(x)dx - n\int_0^1\log(z-x)dx. \qquad (2.60)
\end{aligned}$$

Since

$$\int_0^b \rho(x)dx = 1;$$

see (4.22) in Appendix, it is easily seen that $G(z)=O(|z|^{-1})$ as $z\to\infty$ and $G_+(x)=G_-(x)$ for $x<0$. Furthermore, applying (2.39) and (2.44) to (2.60) implies

$$G_+-G_- = \begin{cases} 0 & : \quad x<a, \\ -2n\widetilde{\phi}_+ = 2n\widetilde{\phi}_- & : \quad a<x<1, \\ -2n\phi_+ = 2n\phi_- & : \quad 1<x<b. \end{cases} \qquad (2.61)$$

Thus, the function $G(z)$ can be analytically continued to $\mathbb{C} \setminus [a, b]$. In terms of $G(z)$, we make the third transformation

$$S(z) := e^{(-nl/2)\sigma_3} R(z) e^{(-G(z)+nl/2)\sigma_3}. \tag{2.62}$$

To compute the jump conditions of $S(z)$, we first state the following lemma.

Lemma 2.11. *Let the functions $a_{12}^{(\pm)}$ and $a_{21}^{(\pm)}$ be defined in (2.14) and (2.15); see also (2.23) and (2.24). For $0 < x < 1$, we have*

$$(a_{21}^{(+)} - a_{21}^{(-)}) e^{-G_+ - G_- + nl} = \frac{e^{n(\widetilde{\phi}_+ + \widetilde{\phi}_-)}}{\widetilde{H}\widetilde{E}^2}. \tag{2.63}$$

For $x > 1$, we have

$$(a_{12}^{(+)} - a_{12}^{(-)}) e^{G_+ + G_- - nl} = \frac{-HE^2}{e^{n(\phi_+ + \phi_-)}}. \tag{2.64}$$

For $z \in \mathbb{C}_\pm$, we have

$$a_{21}^{(\pm)} e^{-2G+nl} = \frac{e^{2n\phi}}{\pm H\widetilde{E}E}, \tag{2.65}$$

$$a_{12}^{(\pm)} e^{2G-nl} = \frac{\mp \widetilde{H}\widetilde{E}E}{e^{2n\widetilde{\phi}}}. \tag{2.66}$$

Proof. Coupling (2.53) and (2.60) gives

$$Ee^{-G} = e^{-ng} \left(\frac{z-1}{z}\right)^{\frac{1-\beta}{2}} \prod_{k=0}^{n-1} (z - X_k), \qquad z \in \mathbb{C}_\pm.$$

Therefore, we have

$$E_+ E_- e^{-(G_+ + G_-)} = e^{-n(g_+ + g_-)} \left(\frac{1-x}{x}\right)^{1-\beta} \prod_{k=0}^{n-1} (x - X_k)^2 \tag{2.67}$$

for $x \in (0, 1)$, and

$$E^2 e^{-(G_+ + G_-)} = e^{-n(g_+ + g_-)} \left(\frac{x-1}{x}\right)^{1-\beta} \prod_{k=0}^{n-1} (x - X_k)^2 \tag{2.68}$$

for $x \in (1, \infty)$, and

$$E^2 e^{-2G} = e^{-2ng} \left(\frac{z-1}{z} \right)^{1-\beta} \prod_{k=0}^{n-1} (z - X_k)^2 \tag{2.69}$$

for $z \in \mathbb{C}_\pm$.

For $x \in (0,1)$, we have from (2.39) and (2.40)

$$g_+ + g_- = v + l - (\phi_+ + \phi_-) = v + l - (\widetilde{\phi}_+ + \widetilde{\phi}_-).$$

Thus, applying (2.56) and (2.57) to (2.67) yields

$$4\widetilde{E}^2 \sin^2(n\pi x - \beta\pi/2) e^{-(G_+ + G_-)} = e^{-n(v+l) + n(\widetilde{\phi}_+ + \widetilde{\phi}_-)} (W/\widetilde{H}) \prod_{k=0}^{n-1} (x - X_k)^2.$$

This equality is same as

$$\frac{4\sin^2(n\pi x - \beta\pi/2)}{W \exp(G_+ + G_- - nv - nl)} \prod_{k=0}^{n-1} (x - X_k)^{-2} = \frac{e^{n(\widetilde{\phi}_+ + \widetilde{\phi}_-)}}{\widetilde{H}\widetilde{E}^2}.$$

Therefore, (2.63) follows from (2.26).

For $x > 1$, we have from (2.40)

$$g_+ + g_- = v + l - (\phi_+ + \phi_-).$$

Thus, applying (2.55) to (2.68) yields

$$E^2 e^{-(G_+ + G_-)} = e^{-n(v+l) + n(\phi_+ + \phi_-)} (E/H) \prod_{k=0}^{n-1} (x - X_k)^2.$$

This equality is same as

$$-W e^{G_+ + G_- - nv - nl} \prod_{k=0}^{n-1} (x - X_k)^2 = \frac{-HE^2}{e^{n(\phi_+ + \phi_-)}}.$$

Therefore, (2.64) follows from (2.27).

For $z \in \mathbb{C}_\pm$, applying (2.40), (2.54) and (2.55) to (2.69) yields

$$\mp 2i \sin(n\pi z - \beta\pi/2) e^{\pm i\pi(nz - \beta/2)} \widetilde{E} E e^{-2G} = e^{-n(v+l) + 2n\phi} (W/H) \prod_{k=0}^{n-1} (z - X_k)^2.$$

This equality is same as

$$\frac{-2i\sin(n\pi z-\beta\pi/2)}{e^{\mp i\pi(nz-\beta/2)}We^{2G-nv-nl}}\prod_{k=0}^{n-1}(z-X_k)^{-2}=\frac{e^{2n\phi}}{\pm H\widetilde{E}E}.$$

Therefore, (2.65) follows from (2.24). Moreover, from (2.39) and (2.59) we have

$$(\widetilde{H}/H)e^{2n(\phi-\widetilde{\phi})}=-e^{\pm 2i\pi(nz-\beta/2)}.$$

Thus, (2.66) follows from (2.25) and (2.65). □

Now, we come back to the transformation (2.62). It is easily seen from (R1) and (2.61) that the matrix-valued function $S(z)$ is analytic in $\mathbb{C}\setminus\Sigma_R$. Let $\Sigma_S:=\Sigma_R$ be the oriented contour depicted in Figure 2.1. We calculate the jump matrices for $S(z)$ in the following proposition.

Proposition 2.12. *On the contour Σ_S, the jump matrix $J_S(z):=S_-(z)^{-1}S_+(z)$ has the following explicit expressions. For $0<x<a$, we have*

$$J_S(x)=\begin{pmatrix}1 & 0\\ \dfrac{e^{2n\widetilde{\phi}}}{\widetilde{H}\widetilde{E}^2} & 1\end{pmatrix}. \tag{2.70}$$

For $a<x<1$, we have

$$J_S(x)=\begin{pmatrix}e^{-2n\widetilde{\phi}_-} & 0\\ \dfrac{1}{\widetilde{H}\widetilde{E}^2} & e^{-2n\widetilde{\phi}_+}\end{pmatrix}. \tag{2.71}$$

For $1<x<b$, we have

$$J_S(x)=\begin{pmatrix}e^{2n\phi_+} & -HE^2\\ 0 & e^{2n\phi_-}\end{pmatrix}. \tag{2.72}$$

For $x>b$, we have

$$J_S(x)=\begin{pmatrix}1 & \dfrac{-HE^2}{e^{2n\phi}}\\ 0 & 1\end{pmatrix}. \tag{2.73}$$

For $z = 1 + i\,\mathrm{Im}\,z$ *with* $\mathrm{Im}\,z \in (0, \pm\delta)$, *we have*

$$J_S(z) = \begin{pmatrix} E/\widetilde{E} & \dfrac{\pm\,\widetilde{H}\widetilde{E}E}{e^{2n\widetilde{\phi}}} \\ \dfrac{e^{2n\phi}}{\pm H\widetilde{E}E} & 1 \end{pmatrix}. \tag{2.74}$$

For $z \in (0, \pm i\delta) \cup (\pm i\delta, 1 \pm i\delta)$, *we have*

$$J_S(z) = \begin{pmatrix} 1 & 0 \\ \dfrac{e^{2n\phi}}{\mp H\widetilde{E}E} & 1 \end{pmatrix}. \tag{2.75}$$

For $z = \mathrm{Re}\,z \pm i\delta$ *with* $\mathrm{Re}\,z \in (1, \infty)$, *we have*

$$J_S(z) = \begin{pmatrix} 1 & \dfrac{\pm\,\widetilde{H}\widetilde{E}E}{e^{2n\widetilde{\phi}}} \\ 0 & 1 \end{pmatrix}. \tag{2.76}$$

The jump conditions of $S(z)$ *on the contour* Σ_S *are illustrated in Figure 2.2.*

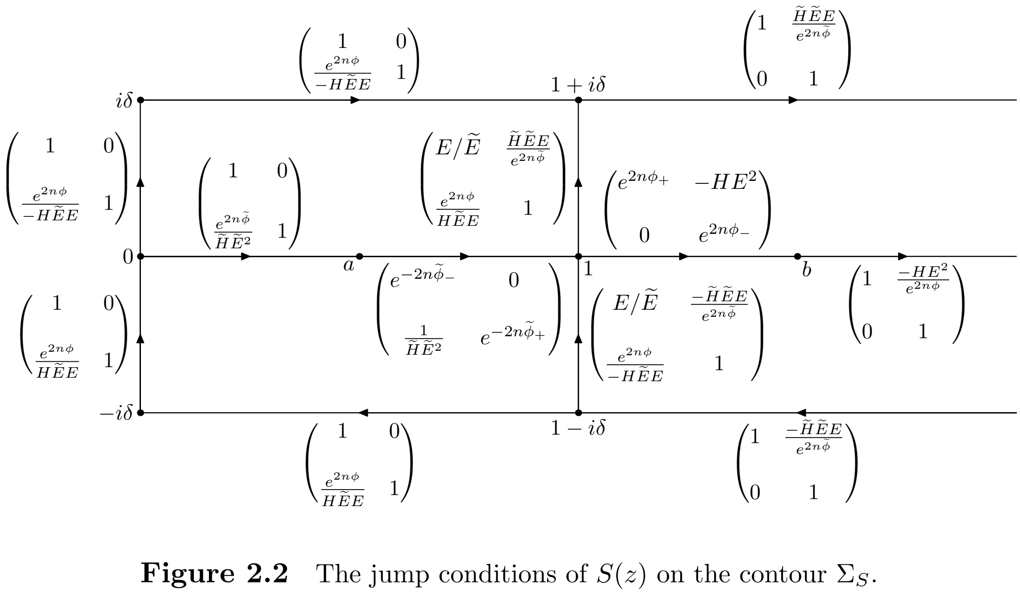

Figure 2.2 The jump conditions of $S(z)$ on the contour Σ_S.

Proof. From (2.62), we have

$$J_S(z) = e^{(G_-(z)-nl/2)\sigma_3} J_R(z) e^{(-G_+(z)+nl/2)\sigma_3}. \tag{2.77}$$

Combining (2.29), (2.63), (2.64) and (2.77) implies

$$J_S(x) = \begin{pmatrix} e^{G_- - G_+} & 0 \\ \dfrac{e^{n(\widetilde{\phi}_+ + \widetilde{\phi}_-)}}{\widetilde{H}\widetilde{E}^2} & e^{G_+ - G_-} \end{pmatrix} \tag{2.78a}$$

for $x \in (0,1)$, and

$$J_S(x) = \begin{pmatrix} e^{G_- - G_+} & \dfrac{-\,HE^2}{e^{n(\phi_+ + \phi_-)}} \\ 0 & e^{G_+ - G_-} \end{pmatrix} \tag{2.78b}$$

for $x \in (1,\infty)$. Applying (2.41), (2.42) and (2.61) to (2.78) gives (2.70)-(2.73) immediately.

Recall that the function $G(z)$ is analytic in $\mathbb{C}\setminus[a,b]$. A combination of (2.28), (2.30), (2.58), (2.65), (2.66) and (2.77) gives (2.74)-(2.76) immediately. $\square$

Proposition 2.13. *The matrix-valued function $S(z)$ defined in (2.62) is the unique solution to the following Riemann-Hilbert problem:*

(S1) $S(z)$ is analytic in $\mathbb{C}\setminus\Sigma_S$;

(S2) for $z \in \Sigma_S$, $S_+(z) = S_-(z)J_S(z)$, where $J_S(z)$ is given in Proposition 2.12;

(S3) for $z \in \mathbb{C}\setminus\Sigma_S$, $S(z) = I + O(|z|^{-1})$ as $z \to \infty$.

Proof. The analyticity condition (S1) is clear from the definition of $S(z)$ in (2.62), and from the analyticity condition (R1) of $R(z)$ in Proposition 2.5. The jump condition (S2) is proved in Proposition 2.12. Furthermore, the normalization condition (R3) of $R(z)$ in Proposition 2.5 gives (S3). The uniqueness is again a direct consequence of Liouville's theorem. $\square$

2.3 The nonlinear steepest-descent method

For $a < x < 1$, we can factorize the jump matrix $J_S(x)$ in (2.71) as below

$$\begin{pmatrix} e^{-2n\widetilde{\phi}_-} & 0 \\ \dfrac{1}{\widetilde{H}\widetilde{E}^2} & e^{-2n\widetilde{\phi}_+} \end{pmatrix} = \begin{pmatrix} \widetilde{E} & \dfrac{\widetilde{H}\widetilde{E}}{e^{2n\widetilde{\phi}_-}} \\ 0 & 1/\widetilde{E} \end{pmatrix} \begin{pmatrix} 0 & -\widetilde{H} \\ 1/\widetilde{H} & 0 \end{pmatrix} \begin{pmatrix} 1/\widetilde{E} & \dfrac{\widetilde{H}\widetilde{E}}{e^{2n\widetilde{\phi}_+}} \\ 0 & \widetilde{E} \end{pmatrix}, \tag{2.79}$$

where we have used (2.42). Similarly, by using (2.41), for $1 < x < b$ we can factorize the jump matrix $J_S(x)$ in (2.72) as below

$$\begin{pmatrix} e^{2n\phi_+} & -HE^2 \\ 0 & e^{2n\phi_-} \end{pmatrix} = \begin{pmatrix} E & 0 \\ \dfrac{e^{2n\phi_-}}{-HE} & 1/E \end{pmatrix} \begin{pmatrix} 0 & -H \\ 1/H & 0 \end{pmatrix} \begin{pmatrix} 1/E & 0 \\ \dfrac{e^{2n\phi_+}}{-HE} & E \end{pmatrix}. \tag{2.80}$$

This suggests the final transformation $S \to T$. Let the domain $\Omega_T = \Omega^1_{T,\pm} \cup \cdots \cup \Omega^4_{T,\pm} \cup \Omega^\infty_T$ and the oriented contour $\Sigma_T = \Sigma^1_{T,\pm} \cup \cdots \cup \Sigma^7_{T,\pm} \cup (0, \infty)$ be as depicted in Figure 2.3.

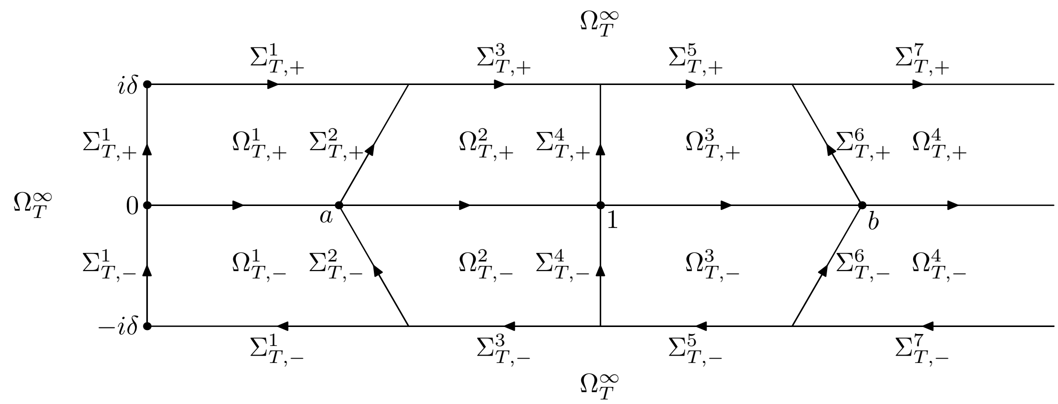

Figure 2.3 The region Ω_T and the contour Σ_T.

We define

$$T(z) := S(z)\widetilde{E}^{\sigma_3} \tag{2.81a}$$

for $z \in \Omega^1_{T,\pm}$, and

$$T(z) := S(z) \begin{pmatrix} \widetilde{E} & \dfrac{\mp \widetilde{H}\widetilde{E}}{e^{2n\widetilde{\phi}}} \\ 0 & 1/\widetilde{E} \end{pmatrix} \tag{2.81b}$$

for $z \in \Omega^2_{T,\pm}$, and

$$T(z) := S(z) \begin{pmatrix} E & 0 \\ \dfrac{e^{2n\phi}}{\pm HE} & 1/E \end{pmatrix} \tag{2.81c}$$

for $z \in \Omega^3_{T,\pm}$, and

$$T(z) := S(z)E^{\sigma_3} \tag{2.81d}$$

for $z \in \Omega^4_{T,\pm} \cup \Omega^\infty_T$. For easy reference, we use Figure 2.4 to illustrate the transformation $S \to T$.

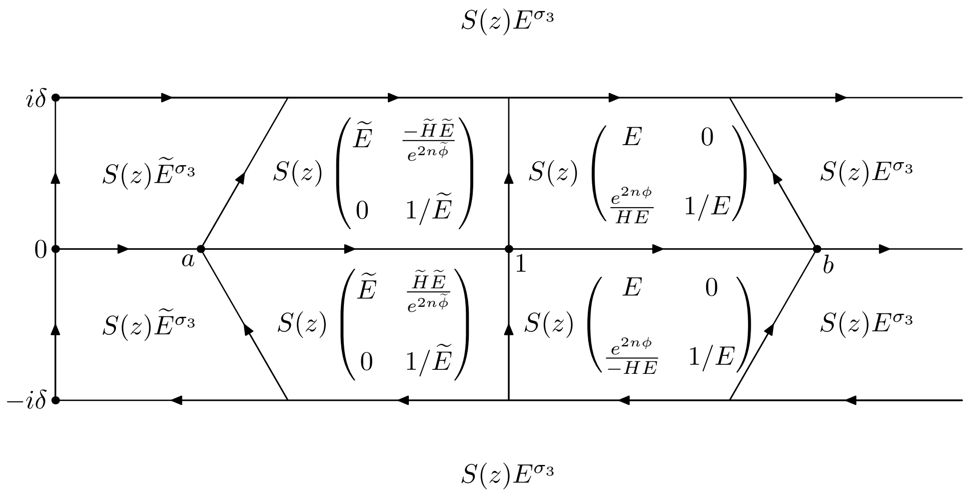

Figure 2.4 The transformation $S \to T$.

We now study the jump conditions of $T(z)$ on the contour Σ_T.

Proposition 2.14. *On the contour Σ_T, the jump matrix $J_T(z) := T_-(z)^{-1}T_+(z)$ can be calculated as below. For $z \in \Sigma^4_{T,\pm}$, we have*

$$J_T(z) = I. \tag{2.82}$$

For $z \in \Sigma^3_{T,\pm}$, we have

$$J_T(z) = \begin{pmatrix} 1 & \dfrac{\pm\widetilde{H}\widetilde{E}}{e^{2n\widetilde{\phi}}E} \\ \dfrac{e^{2n\phi}}{\mp H} & \widetilde{E}/E \end{pmatrix}. \tag{2.83}$$

For $z \in \Sigma^5_{T,\pm}$, we have

$$J_T(z) = \begin{pmatrix} 1 & \dfrac{\pm\widetilde{H}\widetilde{E}}{e^{2n\widetilde{\phi}}E} \\ \dfrac{e^{2n\phi}}{\mp H} & \widetilde{E}/E \end{pmatrix}. \tag{2.84}$$

For $z \in \Sigma^1_{T,\pm}$, we have

$$J_T(z) = \begin{pmatrix} E/\widetilde{E} & 0 \\ \dfrac{e^{2n\phi}}{\mp H} & \widetilde{E}/E \end{pmatrix}. \tag{2.85}$$

For $z \in \Sigma^7_{T,\pm}$, we have

$$J_T(z) = \begin{pmatrix} 1 & \dfrac{\pm\widetilde{H}\widetilde{E}}{e^{2n\widetilde{\phi}}E} \\ 0 & 1 \end{pmatrix}. \tag{2.86}$$

On the positive real line, we have

$$J_T(x) = \begin{pmatrix} 1 & 0 \\ \dfrac{e^{2n\widetilde{\phi}}}{\widetilde{H}} & 1 \end{pmatrix} \tag{2.87a}$$

for $0 < x < a$, and

$$J_T(x) = \begin{pmatrix} 0 & -\widetilde{H} \\ 1/\widetilde{H} & 0 \end{pmatrix} \tag{2.87b}$$

for $a < x < 1$, and

$$J_T(x) = \begin{pmatrix} 0 & -H \\ 1/H & 0 \end{pmatrix} \tag{2.87c}$$

for $1 < x < b$, *and*

$$J_T(x) = \begin{pmatrix} 1 & \dfrac{-H}{e^{2n\phi}} \\ 0 & 1 \end{pmatrix} \tag{2.87d}$$

for $x > b$. *Furthermore, we have*

$$J_T(z) = \begin{pmatrix} 1 & \dfrac{\pm \widetilde{H}}{e^{2n\widetilde{\phi}}} \\ 0 & 1 \end{pmatrix} \tag{2.88a}$$

for $z \in \Sigma^2_{T,\pm}$, *and*

$$J_T(z) = \begin{pmatrix} 1 & 0 \\ \dfrac{e^{2n\phi}}{\pm H} & 1 \end{pmatrix} \tag{2.88b}$$

for $z \in \Sigma^6_{T,\pm}$. *The jump conditions of* $T(z)$ *on the contour* Σ_T *are illustrated in Figure 2.5.*

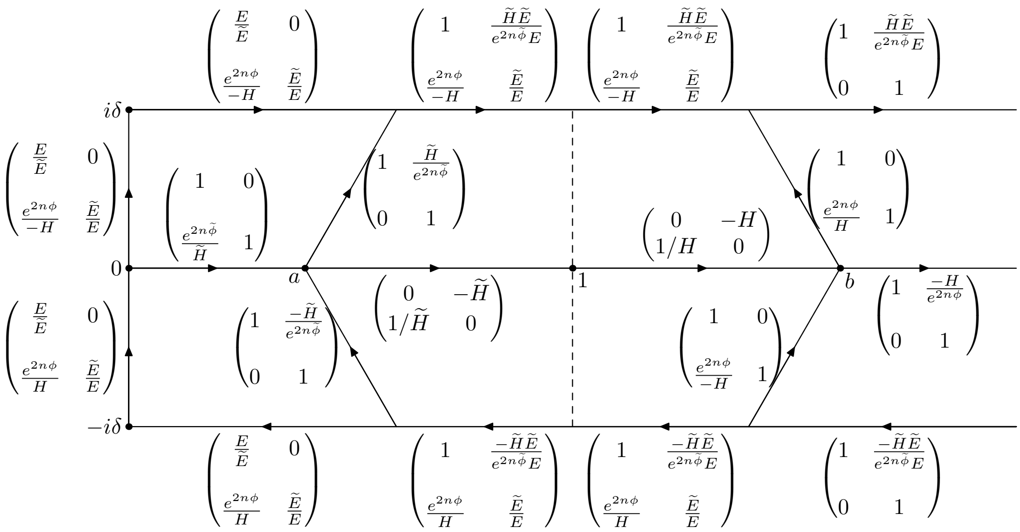

Figure 2.5 The jump conditions of $T(z)$. The dashed line means that there is actually no jump on this line.

Proof. For $z \in \Sigma^4_{T,\pm}$, we obtain from (2.74) and (2.81)

$$J_T(z) = \begin{pmatrix} E & 0 \\ \dfrac{e^{2n\phi}}{\pm HE} & 1/E \end{pmatrix}^{-1} \begin{pmatrix} E/\widetilde{E} & \dfrac{\pm \widetilde{H}\widetilde{E}E}{e^{2n\widetilde{\phi}}} \\ \dfrac{e^{2n\phi}}{\pm H\widetilde{E}E} & 1 \end{pmatrix} \begin{pmatrix} \widetilde{E} & \dfrac{\mp \widetilde{H}\widetilde{E}}{e^{2n\widetilde{\phi}}} \\ 0 & 1/\widetilde{E} \end{pmatrix}. \tag{2.89}$$

A combination of (2.39), (2.58) and (2.59) gives

$$(\widetilde{H}/H)e^{2n(\phi-\widetilde{\phi})} = -e^{\pm 2i\pi(nz-\beta/2)} = E/\widetilde{E} - 1.$$

This equality can be rewritten as

$$\frac{-\, e^{2n(\phi-\widetilde{\phi})}\widetilde{H}}{HE} + \frac{1}{\widetilde{E}} = \frac{1}{E}. \tag{2.90}$$

Therefore, we have

$$\begin{pmatrix} E/\widetilde{E} & \dfrac{\pm \widetilde{H}\widetilde{E}E}{e^{2n\widetilde{\phi}}} \\ \dfrac{e^{2n\phi}}{\pm H\widetilde{E}E} & 1 \end{pmatrix} \begin{pmatrix} \widetilde{E} & \dfrac{\mp \widetilde{H}\widetilde{E}}{e^{2n\widetilde{\phi}}} \\ 0 & 1/\widetilde{E} \end{pmatrix} = \begin{pmatrix} E & 0 \\ \dfrac{e^{2n\phi}}{\pm HE} & 1/E \end{pmatrix}. \tag{2.91}$$

Coupling (2.89) and (2.91) yields (2.82).

For $z \in \Sigma^3_{T,\pm}$, by applying (2.75) to (2.81) we obtain

$$J_T(z) = \begin{pmatrix} \widetilde{E} & \dfrac{\mp \widetilde{H}\widetilde{E}}{e^{2n\widetilde{\phi}}} \\ 0 & 1/\widetilde{E} \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ \dfrac{e^{2n\phi}}{\mp H\widetilde{E}E} & 1 \end{pmatrix} E(z)^{\sigma_3}.$$

On account of (2.90), we have

$$J_T(z) = \begin{pmatrix} 1 & \dfrac{\pm \widetilde{H}\widetilde{E}}{e^{2n\widetilde{\phi}}E} \\ \dfrac{e^{2n\phi}}{\mp H} & \widetilde{E}/E \end{pmatrix}.$$

Thus, (2.83) is proved. Similarly, for $z \in \Sigma^5_{T,\pm}$, we have from (2.76) and (2.81)

$$J_T(z) = \begin{pmatrix} E & 0 \\ \dfrac{e^{2n\phi}}{\pm HE} & 1/E \end{pmatrix}^{-1} \begin{pmatrix} 1 & \dfrac{\pm \widetilde{H}\widetilde{E}E}{e^{2n\widetilde{\phi}}} \\ 0 & 1 \end{pmatrix} E(z)^{\sigma_3}$$

Applying (2.90) to the last equation yields (2.84).

For $z \in \Sigma^1_{T,\pm}$, we have from (2.75) and (2.81)

$$J_T(z) = \widetilde{E}(z)^{-\sigma_3} \begin{pmatrix} 1 & 0 \\ \dfrac{e^{2n\phi}}{\mp H\widetilde{E}E} & 1 \end{pmatrix} E(z)^{\sigma_3} = \begin{pmatrix} E/\widetilde{E} & 0 \\ \dfrac{e^{2n\phi}}{\mp H} & \widetilde{E}/E \end{pmatrix}.$$

This proves (2.85). Similarly, for $z \in \Sigma^7_{T,\pm}$, we have from (2.76) and (2.81)

$$J_T(z) = E(z)^{-\sigma_3} \begin{pmatrix} 1 & \dfrac{\pm \widetilde{H}\widetilde{E}E}{e^{2n\widetilde{\phi}}} \\ 0 & 1 \end{pmatrix} E(z)^{\sigma_3} = \begin{pmatrix} 1 & \dfrac{\pm \widetilde{H}\widetilde{E}}{e^{2n\widetilde{\phi}}E} \\ 0 & 1 \end{pmatrix}.$$

This gives (2.86).

For $0 < x < a$, we obtain from (2.70) and (2.81)

$$J_T(x) = \widetilde{E}(x)^{-\sigma_3} \begin{pmatrix} 1 & 0 \\ \dfrac{e^{2n\widetilde{\phi}}}{\widetilde{H}\widetilde{E}^2} & 1 \end{pmatrix} \widetilde{E}(x)^{\sigma_3} = \begin{pmatrix} 1 & 0 \\ \dfrac{e^{2n\widetilde{\phi}}}{\widetilde{H}} & 1 \end{pmatrix}.$$

Similarly, for $x > b$, we obtain from (2.73) and (2.81)

$$J_T(x) = E(x)^{-\sigma_3} \begin{pmatrix} 1 & \dfrac{-HE^2}{e^{2n\phi}} \\ 0 & 1 \end{pmatrix} E(x)^{\sigma_3} = \begin{pmatrix} 1 & \dfrac{-H}{e^{2n\phi}} \\ 0 & 1 \end{pmatrix}.$$

Thus, (2.87a) and (2.87d) are proved.

For $a < x < 1$, by applying (2.71) to (2.81) we obtain

$$J_T(x) = \begin{pmatrix} \widetilde{E} & \dfrac{\widetilde{H}\widetilde{E}}{e^{2n\widetilde{\phi}_-}} \\ 0 & 1/\widetilde{E} \end{pmatrix}^{-1} \begin{pmatrix} e^{-2n\widetilde{\phi}_-} & 0 \\ \dfrac{1}{\widetilde{H}\widetilde{E}^2} & e^{-2n\widetilde{\phi}_-} \end{pmatrix} \begin{pmatrix} \widetilde{E} & \dfrac{-\widetilde{H}\widetilde{E}}{e^{2n\widetilde{\phi}_+}} \\ 0 & 1/\widetilde{E} \end{pmatrix}.$$

This together with (2.79) gives (2.87b). Similarly, for $1 < x < b$, by applying (2.72) to (2.81) we obtain

$$J_T(x) = \begin{pmatrix} E & 0 \\ \dfrac{e^{2n\phi_-}}{-HE} & 1/E \end{pmatrix}^{-1} \begin{pmatrix} e^{2n\phi_+} & -HE^2 \\ 0 & e^{2n\phi_-} \end{pmatrix} \begin{pmatrix} E & 0 \\ \dfrac{e^{2n\phi_+}}{HE} & 1/E \end{pmatrix}.$$

Thus, (2.87c) follows from (2.80).

Finally, since $S(z)$ has no jump on $\Sigma_{T,\pm}^2$ and $\Sigma_{T,\pm}^6$, we obtain (2.88) from the definition of $T(z)$ in (2.81). This ends the proof of the proposition. □

Proposition 2.15. *The matrix-valued function $T(z)$ defined in (2.81) is the unique solution to the following Riemann-Hilbert problem:*

(T1) $T(z)$ is analytic in $\mathbb{C} \setminus \Sigma_T$;

(T2) for $z \in \Sigma_T$, $T_+(z) = T_-(z)J_T(z)$, where $J_T(z)$ is given in Proposition 2.14;

(T3) for $z \in \mathbb{C} \setminus \Sigma_T$, $T(z) = I + O(|z|^{-1})$ as $z \to \infty$.

Proof. The analyticity follows from (S1) in Proposition 2.13 and the definition of $T(z)$. Proposition 2.14 gives (T2). Furthermore, (S3) in Proposition 2.13 yields (T3). The uniqueness is again an immediate consequence of Liouville's theorem. □

With the aid of Figure 2.5, we observe from (2.58) and Propositions 2.8 & 2.14 that as $n \to \infty$, the jump matrix $J_T(z)$ converges exponentially fast to the identity for z bounded away from $[a, b] \cup \{0\}$. The limiting Riemann-Hilbert problem can be divided into several local problems, whose solutions can be constructed explicitly. Since these solutions to the local Riemann-Hilbert problems are not unique, we shall choose as in [7] some specific ones, which are asymptotically equal to each other in the overlapping regions. By piecing them together, we build a function that is defined in the whole complex plane. This matrix-valued function is our desired parametrix.

We first consider the Riemann-Hilbert problem:

(M1) $M(z)$ is analytic in $\mathbb{C} \setminus [a, b]$;

(M2) $M(z)$ satisfies the following jump conditions

$$M_+(x) = M_-(x) \begin{pmatrix} 0 & -\widetilde{H} \\ 1/\widetilde{H} & 0 \end{pmatrix} \tag{2.92a}$$

for $a < x < 1$, and

$$M_+(x) = M_-(x) \begin{pmatrix} 0 & -H \\ 1/H & 0 \end{pmatrix} \tag{2.92b}$$

for $1 < x < b$;

(M3) $M(z) = I + O(|z|^{-1})$, as $z \to \infty$.

Recall that $H(z) = [z/(z-1)]^{1-\beta} W(z)$ and $\widetilde{H}(z) = [z/(1-z)]^{1-\beta} W(z)$, where

$$W(z) = \frac{2ni\pi \Gamma(nz + \beta/2) c^{-\beta/2}}{\Gamma(nz + 1 - \beta/2)};$$

see (2.22), (2.55) and (2.56). Define

$$V(z) := \log \frac{\Gamma(nz + 1 - \beta/2)}{z^{1-\beta} \Gamma(nz + \beta/2)} - \log(2ni\pi c^{-\beta/2}). \tag{2.93}$$

Clearly,

$$H(z) = (z-1)^{\beta-1} e^{-V(z)}, \qquad \widetilde{H}(z) = (1-z)^{\beta-1} e^{-V(z)}. \tag{2.94}$$

From the Stirling series [1, (6.1.40) and (6.3.18)], we have

$$\log \Gamma(z) = (z - \frac{1}{2}) \log z - z + \frac{1}{2} \log(2\pi) + O(|z|^{-1}), \qquad \frac{\Gamma'(z)}{\Gamma(z)} = \log z - \frac{1}{2z} + O(|z|^{-2})$$

as $z \to \infty$. The estimate holds uniformly for z bounded away from the negative real line. Thus, we obtain the double asymptotic behavior for $V(z)$ as $n \to \infty$ or $z \to \infty$,

$$V(z) = -\beta \log n - \log(2i\pi c^{-\beta/2}) + O(\frac{1}{n|z|}), \qquad V'(z) = O(\frac{1}{n|z|^2}), \tag{2.95}$$

which again holds uniformly for z bounded away from the negative real line. For z bounded away from $(-\infty, 0] \cup \{1\}$, it follows from (2.94) and (2.95) that

$$|n^{-\beta}H(z)| + |n^{\beta}H(z)^{-1}| + |n^{-\beta}\widetilde{H}(z)| + |n^{\beta}\widetilde{H}(z)^{-1}| = O(1) \tag{2.96}$$

as $n \to \infty$. Furthermore, for $\operatorname{Re} z \geq 0$, we have from (2.22) and Stirling's formula that $W(z)^{-1}$ is uniformly bounded as $n \to \infty$. Thus, from (2.55) and (2.56), we obtain

$$|H(z)^{-1}| + |\widetilde{H}(z)^{-1}| = O(1) \tag{2.97}$$

uniformly for $\operatorname{Re} z \geq 0$ and $z \neq 1$. Here, we have used the assumption $1 \leq \beta < 2$. Now, we introduce the function

$$\widetilde{G}(z) := -\int_z^{\infty}\int_a^b \frac{V'(s)\sqrt{(s-a)(b-s)}}{2\pi(s-\zeta)\sqrt{(\zeta-a)(\zeta-b)}}dsd\zeta. \tag{2.98}$$

Lemma 2.16. *The function $\widetilde{G}(z)$ defined in (2.98) is a solution to the Riemann-Hilbert problem:*

(G1) $\widetilde{G}(z)$ is analytic in $\mathbb{C} \setminus [a, b]$;

(G2) for $x \in (a, b)$, $\widetilde{G}(z)$ satisfies the jump condition

$$\widetilde{G}_+(x) + \widetilde{G}_-(x) - V(x) - L = 0, \tag{2.99}$$

where $L := 2\widetilde{G}(b) - V(b)$ is a constant independent of x;

(G3) $\widetilde{G}(z) = O(|z|^{-1})$, as $z \to \infty$.

As $n \to \infty$, we have

$$\widetilde{G}(z) = O(1/n) \tag{2.100}$$

uniformly for $z \in \mathbb{C}$. Here, the value of $\widetilde{G}(x)$ at $x \in (a, b)$ takes the meaning of boundary value from the upper or lower half-plane. Therefore (2.100) implies that $|\widetilde{G}_+(x)| + |\widetilde{G}_-(x)| = O(1/n)$ for $x \in (a, b)$. Furthermore, we have following asymptotic formula for the constant

$$L := 2\widetilde{G}(b) - V(b) = \beta \log n + \log(2i\pi c^{-\beta/2}) + O(1/n). \tag{2.101}$$

Proof. From (2.98), we obtain

$$\widetilde{G}'(z) = \int_a^b \frac{V'(s)\sqrt{(s-a)(b-s)}ds}{2\pi(s-z)\sqrt{(z-a)(z-b)}}. \tag{2.102}$$

It is easily seen that $\widetilde{G}'(z)$ is analytic in $\mathbb{C}\setminus[a,b]$ and $\widetilde{G}'_+(x)+\widetilde{G}'_-(x) = V'(x)$ for $x \in (a,b)$. Moreover, $\widetilde{G}'(z) = O(|z|^{-2})$ as $z \to \infty$. Thus, (G1)-(G3) follows.

From (2.95) and (2.102), we have $(1+|z|^2)|\widetilde{G}'(z)| = O(1/n)$ as $n \to \infty$. This estimate is uniform for $z \in \mathbb{C}$. Therefore, $\widetilde{G}(z) = O(1/n)$, thus giving (2.100).

Finally, (2.101) follows from (2.95) and (2.100). □

With the aid of the function $\widetilde{G}(z)$, we now solve the Riemann-Hilbert problem (M1)-(M3) explicitly.

Proposition 2.17. *The Riemann-Hilbert problem (M1)-(M3) has a solution given by*

$$M(z) = \begin{pmatrix} \dfrac{(z-1)^{\frac{1-\beta}{2}}(\frac{\sqrt{z-a}+\sqrt{z-b}}{2})^{\beta}}{(z-a)^{1/4}(z-b)^{1/4}e^{-\widetilde{G}(z)}} & \dfrac{-i(z-1)^{\frac{\beta-1}{2}}(\frac{\sqrt{z-a}-\sqrt{z-b}}{2})^{\beta}}{(z-a)^{1/4}(z-b)^{1/4}e^{\widetilde{G}(z)-L}} \\[2ex] \dfrac{i(z-1)^{\frac{1-\beta}{2}}(\frac{\sqrt{z-a}-\sqrt{z-b}}{2})^{2-\beta}}{(z-a)^{1/4}(z-b)^{1/4}e^{L-\widetilde{G}(z)}} & \dfrac{(z-1)^{\frac{\beta-1}{2}}(\frac{\sqrt{z-a}+\sqrt{z-b}}{2})^{2-\beta}}{(z-a)^{1/4}(z-b)^{1/4}e^{\widetilde{G}(z)}} \end{pmatrix}. \tag{2.103}$$

Proof. Since $\widetilde{G}(z)$ is analytic in $\mathbb{C}\setminus[a,b]$, the entries of $M(z)$ can be analytically continued to the interval $(-\infty,a)$. Thus, (M1) follows.

The jump conditions in (M2) can be verified as below. For $x \in (1,b)$, we obtain from (2.94) and (2.103) that

$$M^{11}_{\pm}(x) = \frac{(x-1)^{\frac{1-\beta}{2}}(\frac{\sqrt{x-a}\pm i\sqrt{b-x}}{2})^{\beta}}{(x-a)^{1/4}(b-x)^{1/4}e^{\pm i\pi/4}e^{-\widetilde{G}_{\pm}(x)}},$$

$$M^{12}_{\mp}(x) = \frac{-iH(x)(x-1)^{\frac{1-\beta}{2}}(\frac{\sqrt{x-a}\pm i\sqrt{b-x}}{2})^{\beta}}{(x-a)^{1/4}(b-x)^{1/4}e^{\mp i\pi/4}e^{\widetilde{G}_{\mp}(x)-V(x)-L}}.$$

Thus, the relation (2.99) implies that $M^{12}_{\mp}(x)/M^{11}_{\pm}(x) = \pm H(x)$ for $x \in (1, b)$. On the other hand, for $x \in (a, 1)$, we have from (2.94) and (2.103)

$$M^{11}_{\pm}(x) = \frac{(1-x)^{\frac{1-\beta}{2}} e^{\frac{\pm i\pi(1-\beta)}{2}} (\frac{\sqrt{x-a} \pm i\sqrt{b-x}}{2})^{\beta}}{(x-a)^{1/4}(b-x)^{1/4} e^{\pm i\pi/4} e^{-\widetilde{G}_{\pm}(x)}},$$
$$M^{12}_{\mp}(x) = \frac{-i\widetilde{H}(x)(1-x)^{\frac{1-\beta}{2}} e^{\frac{\pm i\pi(1-\beta)}{2}} (\frac{\sqrt{x-a} \pm i\sqrt{b-x}}{2})^{\beta}}{(x-a)^{1/4}(b-x)^{1/4} e^{\mp i\pi/4} e^{\widetilde{G}_{\mp}(x) - V(x) - L}}.$$

Coupling this with (2.99) yields $M^{12}_{\mp}(x)/M^{11}_{\pm}(x) = \pm \widetilde{H}(x)$ for $x \in (a, 1)$. Similarly, a combination of (2.94), (2.99) and (2.103) gives

$$\frac{M^{22}_{\mp}(x)}{M^{21}_{\pm}(x)} = \begin{cases} \pm H(x), & x \in (1, b), \\ \pm \widetilde{H}(x), & x \in (a, 1). \end{cases}$$

This proves (M2).

By (G3) in Lemma 2.16, we have $\widetilde{G}(z) = O(|z|^{-1})$ as $z \to \infty$. Hence, it is easily seen from (2.103) that $M(z) = I + O(|z|^{-1})$ as $z \to \infty$. □

From (2.95), (2.100) and (2.101) we have, as $n \to \infty$, $|\widetilde{G}(z)| + |V(z) + L| = O(1/n)$ uniformly for z bounded away from the negative real line. By virtue of the relations

$$\sqrt{z-a} + \sqrt{z-b} = e^{\pm i\pi/2}(\sqrt{b-z} + \sqrt{a-z}),$$
$$\sqrt{z-a} - \sqrt{z-b} = e^{\mp i\pi/2}(\sqrt{b-z} - \sqrt{a-z}),$$

we obtain from (2.94) and (2.103) that

$$\widetilde{H}^{-\sigma_3/2} M \widetilde{H}^{\sigma_3/2}$$
$$= \begin{pmatrix} \frac{(1-z)^{\frac{1-\beta}{2}} (\frac{\sqrt{b-z}+\sqrt{a-z}}{2})^{\beta}}{(b-z)^{1/4}(a-z)^{1/4}} & \frac{i(1-z)^{\frac{1-\beta}{2}} (\frac{\sqrt{b-z}-\sqrt{a-z}}{2})^{\beta}}{(b-z)^{1/4}(a-z)^{1/4}} \\ \frac{-i(1-z)^{\frac{\beta-1}{2}} (\frac{\sqrt{b-z}-\sqrt{a-z}}{2})^{2-\beta}}{(b-z)^{1/4}(a-z)^{1/4}} & \frac{(1-z)^{\frac{\beta-1}{2}} (\frac{\sqrt{b-z}+\sqrt{a-z}}{2})^{2-\beta}}{(b-z)^{1/4}(a-z)^{1/4}} \end{pmatrix}$$
$$\times \left(I + O(\frac{1}{n})\right),$$

which holds uniformly for z bounded away from the negative real line. Define

$$\widetilde{m}(z) := \frac{(1-z)^{\frac{1-\beta}{2}\sigma_3}}{(b-z)^{1/4}(a-z)^{1/4}} \begin{pmatrix} (\frac{\sqrt{b-z}+\sqrt{a-z}}{2})^{\beta} & i(\frac{\sqrt{b-z}-\sqrt{a-z}}{2})^{\beta} \\ -i(\frac{\sqrt{b-z}-\sqrt{a-z}}{2})^{2-\beta} & (\frac{\sqrt{b-z}+\sqrt{a-z}}{2})^{2-\beta} \end{pmatrix} \times \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \tag{2.104}$$

As $n \to \infty$, we have

$$\widetilde{H}(z)^{-\sigma_3/2} M(z) \widetilde{H}(z)^{\sigma_3/2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \widetilde{m}(z) \left(I + O(\frac{1}{n}) \right). \tag{2.105}$$

Similarly, define

$$m(z) := \frac{(z-1)^{\frac{1-\beta}{2}\sigma_3}}{(z-a)^{1/4}(z-b)^{1/4}} \begin{pmatrix} (\frac{\sqrt{z-a}+\sqrt{z-b}}{2})^{\beta} & -i(\frac{\sqrt{z-a}-\sqrt{z-b}}{2})^{\beta} \\ i(\frac{\sqrt{z-a}-\sqrt{z-b}}{2})^{2-\beta} & (\frac{\sqrt{z-a}+\sqrt{z-b}}{2})^{2-\beta} \end{pmatrix} \times \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \tag{2.106}$$

From (2.94) and (2.103), we obtain

$$H(z)^{-\sigma_3/2} M(z) H(z)^{\sigma_3/2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = m(z) \left(I + O(\frac{1}{n}) \right) \tag{2.107}$$

as $n \to \infty$. The estimates (2.105) and (2.107) hold uniformly for z bounded away from the negative real line. Recall that we are using capital letters to emphasize the dependence on n; see the paragraph before Proposition 2.1. The small letters $\widetilde{m}$ and m in (2.104) and (2.106), respectively, indicate that these two matrix-valued functions are independent of n. We would also like to emphasize that for any small $\varepsilon > 0$, the matrix-valued function $m(z)(z-b)^{\sigma_3/4}$ is analytic in $U(b,\varepsilon) := \{z \in \mathbb{C} : |z-b| < \varepsilon\}$, and the matrix-valued function $\widetilde{m}(z)(a-z)^{-\sigma_3/4}$ is analytic in $U(a,\varepsilon) := \{z \in \mathbb{C} : |z-a| < \varepsilon\}$.

Next, we find the solution to the scalar Riemann-Hilbert problem:

(D1) $D(z)$ is analytic in $\mathbb{C} \setminus (-i\infty, i\infty)$;

(D2) $D(z)$ satisfies the jump condition

$$D_+(z) = D_-(z)\frac{E(z)}{\widetilde{E}(z)}, \qquad z \in (-i\infty, i\infty), \tag{2.108}$$

where the functions $D_+(z)$ and $D_-(z)$ denote the boundary values of $D(z)$ taken from the left and right of the imaginary line respectively;

(D3) for $z \in \mathbb{C} \setminus (-i\infty, i\infty)$, $D(z) = 1 + O(|z|^{-1})$ as $z \to \infty$.

Recall from (2.58) that $E(z)/\widetilde{E}(z) = 1 - e^{\pm 2i\pi(nz-\beta/2)}$. The solution to the Riemann-Hilbert problem (D1)-(D3) is given by

$$\begin{aligned} D(z) &= \exp\left\{\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \log\left(\frac{E(\zeta)}{\widetilde{E}(\zeta)}\right)\frac{d\zeta}{\zeta - z}\right\} \\ &= \exp\left\{\frac{1}{2\pi i}\int_0^{\infty}\left[\frac{\log(1-e^{-2n\pi s - i\pi\beta})}{s+iz} - \frac{\log(1-e^{-2n\pi s + i\pi\beta})}{s-iz}\right] ds\right\}. \end{aligned} \tag{2.109}$$

It can be shown that as $n \to \infty$, the function $D(z)$ converges uniformly to the constant "1" for z bounded away from the origin; see Section 4.3 in Appendix.

We now introduce the so-called *Airy parametrix* defined by

$$A(z) := \begin{pmatrix} \operatorname{Ai}(z) & \omega^2 \operatorname{Ai}(\omega^2 z) \\ i \operatorname{Ai}'(z) & i\omega \operatorname{Ai}'(\omega^2 z) \end{pmatrix} \tag{2.110a}$$

for $\arg z \in (0, 2\pi/3)$, and

$$A(z) := \begin{pmatrix} -\omega \operatorname{Ai}(\omega z) & \omega^2 \operatorname{Ai}(\omega^2 z) \\ -i\omega^2 \operatorname{Ai}'(\omega z) & i\omega \operatorname{Ai}'(\omega^2 z) \end{pmatrix} \tag{2.110b}$$

for $\arg z \in (2\pi/3, \pi)$, and

$$A(z) := \begin{pmatrix} -\omega^2 \operatorname{Ai}(\omega^2 z) & -\omega \operatorname{Ai}(\omega z) \\ -i\omega \operatorname{Ai}'(\omega^2 z) & -i\omega^2 \operatorname{Ai}'(\omega z) \end{pmatrix} \tag{2.110c}$$

for $\arg z \in (-\pi, -2\pi/3)$, and

$$A(z) := \begin{pmatrix} \operatorname{Ai}(z) & -\omega \operatorname{Ai}(\omega z) \\ i \operatorname{Ai}'(z) & -i\omega^2 \operatorname{Ai}'(\omega z) \end{pmatrix} \tag{2.110d}$$

for $\arg z \in (-2\pi/3, 0)$. By virtue of the identity of the Airy function $\mathrm{Ai}(z) + \omega \mathrm{Ai}(\omega z) + \omega^2 \mathrm{Ai}(\omega^2 z) = 0$, the Airy parametrix defined in (2.110) has the jump conditions:

$$A_+(z) = A_-(z) \begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix} \tag{2.111a}$$

for $z \in (0, \infty e^{\pm 2\pi/3})$, and

$$A_+(z) = A_-(z) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.111b}$$

for $z \in (-\infty, 0)$, and

$$A_+(z) = A_-(z) \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \tag{2.111c}$$

for $z \in (0, \infty)$. The Airy parametrix and its jump conditions are illustrated in Figure 2.6.

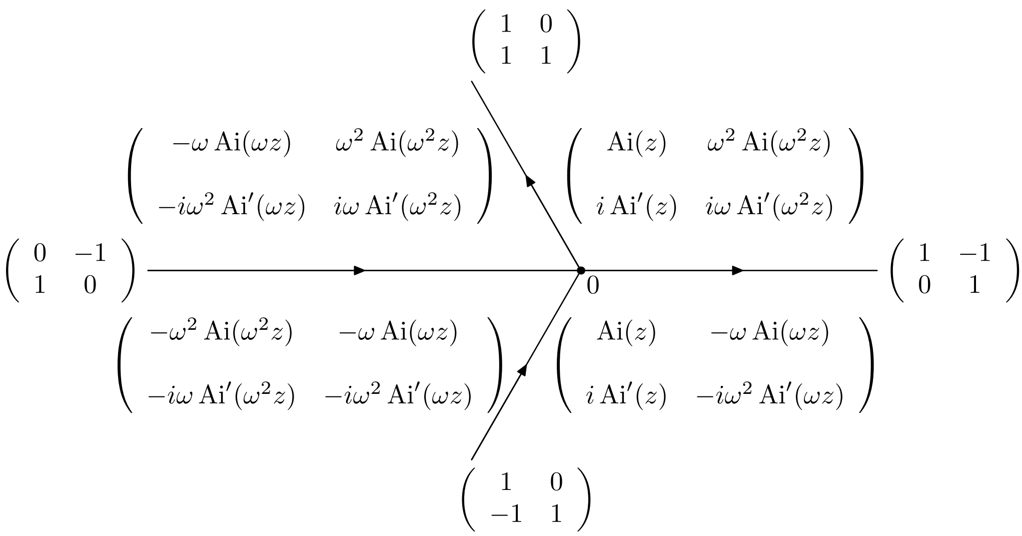

Figure 2.6 The Airy parametrix and its jump conditions.

Recall the asymptotic expansions of the Airy function and its derivative (cf. [22, p. 392] or [28, p. 47])

$$\mathrm{Ai}(z) \sim \frac{z^{-1/4}}{2\sqrt{\pi}} e^{-\frac{2}{3}z^{3/2}} \sum_{s=0}^{\infty} \frac{(-1)^s u_s}{(\frac{2}{3}z^{3/2})^s}, \qquad \mathrm{Ai}'(z) \sim -\frac{z^{1/4}}{2\sqrt{\pi}} e^{-\frac{2}{3}z^{3/2}} \sum_{s=0}^{\infty} \frac{(-1)^s v_s}{(\frac{2}{3}z^{3/2})^s} \tag{2.112}$$

as $z \to \infty$ with $|\arg z| < \pi$, where u_s and v_s are constants with $u_0 = v_0 = 1$. Therefore, by applying (2.112) to (2.110), we obtain

$$A(z) = \frac{z^{-\sigma_3/4}}{2\sqrt{\pi}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} (I + O(|z|^{-3/2}))e^{-\frac{2}{3}z^{3/2}\sigma_3}, \qquad z \to \infty. \tag{2.113}$$

Finally, we construct the parametrix $T_{par}(z)$. Let δ_0 be determined in Remark 2.9. Fix any $0 < \varepsilon < \delta < \delta_0$ and denote by $U(z_0, \varepsilon)$ the open disk centered at z_0 with radius ε, where $z_0 = 0$, a or b. We define

$$T_{par}(z) := M(z) \tag{2.114}$$

for $z \in \mathbb{C} \setminus (U(0,\varepsilon) \cup U(a,\varepsilon) \cup U(b,\varepsilon))$, and

$$T_{par}(z) := M(z)D(z)^{\sigma_3} \tag{2.115}$$

for $z \in U(0,\varepsilon)$, and

$$T_{par}(z) := \sqrt{\pi} H(z)^{\sigma_3/2} m(z) F(z)^{\sigma_3/4} A(F(z)) e^{n\phi(z)\sigma_3} H(z)^{-\sigma_3/2} \tag{2.116}$$

for $z \in U(b,\varepsilon)$, and

$$T_{par}(z) := \sqrt{\pi} \widetilde{H}(z)^{\sigma_3/2} \widetilde{m}(z) \widetilde{F}(z)^{-\sigma_3/4} \sigma_1 A(\widetilde{F}(z)) \sigma_1 e^{n\widetilde{\phi}(z)\sigma_3} \widetilde{H}(z)^{-\sigma_3/2} \tag{2.117}$$

for $z \in U(a,\varepsilon)$, where the functions $F(z)$ and $\widetilde{F}(z)$ are defined by

$$F(z) := \left(\frac{3}{2} n\phi(z)\right)^{2/3}, \qquad \widetilde{F}(z) := \left(-\frac{3}{2} n\widetilde{\phi}(z)\right)^{2/3}, \tag{2.118}$$

and $\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are Pauli matrices.

Remark 2.18. *Now, we determine the precise shape of the curves $\Sigma^2_{T,\pm}$ and $\Sigma^6_{T,\pm}$ in Figure 2.3. Recall the definition of the functions F and $\widetilde{F}$ in (2.118). It follows from (2.45) and (2.46) that*

$$F(z) \sim \left(\frac{2n}{b\sqrt{b-a}}\right)^{2/3} (z - b) \tag{2.119}$$

as $z \to b$, and

$$\widetilde{F}(z) \sim \left(\frac{2n}{a\sqrt{b-a}}\right)^{2/3}(a-z) \tag{2.120}$$

as $z \to a$. Furthermore, the function $F(z)$ is analytic in $U(b,\delta_0)$ and the function $\widetilde{F}(z)$ is analytic in $U(a,\delta_0)$; see the choice of δ_0 in Remark 2.9. We choose $\Sigma_{T,\pm}^6$ to be the inverse image of the rays $(0,\infty e^{\pm 2\pi/3})$ under the holomorphic map F, and $\Sigma_{T,\pm}^2$ to be the inverse image of the rays $(0,\infty e^{\mp 2\pi/3})$ under the holomorphic map $\widetilde{F}$.

Define

$$K(z) := n^{-\beta\sigma_3/2}T(z)T_{par}^{-1}(z)n^{\beta\sigma_3/2}. \tag{2.121}$$

The jump conditions of the function $K(z)$ are studied in the following proposition.

Proposition 2.19. *Let Σ_K be the contour shown in Figure 2.7. The matrix-valued function $K(z)$ is analytic in $\mathbb{C}\setminus\Sigma_K$. On the contour Σ_K, the jump matrix $J_K(z) := K_-(z)^{-1}K_+(z)$ has the following explicit expressions. For $z \in \Sigma_{K,\pm}^1$, we have*

$$J_K(z) = n^{-\beta\sigma_3/2}M\begin{pmatrix} E/\widetilde{E} & 0 \\ \dfrac{e^{2n\phi}}{\mp H} & \widetilde{E}/E \end{pmatrix}M^{-1}n^{\beta\sigma_3/2}. \tag{2.122}$$

For $z \in \Sigma_{K,\pm}^2$, we have

$$J_K(z) = n^{-\beta\sigma_3/2}M\begin{pmatrix} 1 & \dfrac{\pm\widetilde{H}}{e^{2n\widetilde{\phi}}} \\ 0 & 1 \end{pmatrix}M^{-1}n^{\beta\sigma_3/2}. \tag{2.123}$$

For $z \in \Sigma_{K,\pm}^3 \cup \Sigma_{K,\pm}^5$, we have

$$J_K(z) = n^{-\beta\sigma_3/2}M\begin{pmatrix} 1 & \dfrac{\pm\widetilde{H}\widetilde{E}}{e^{2n\widetilde{\phi}}E} \\ \dfrac{e^{2n\phi}}{\mp H} & \widetilde{E}/E \end{pmatrix}M^{-1}n^{\beta\sigma_3/2}. \tag{2.124}$$

For $z \in \Sigma^6_{K,\pm}$, we have

$$J_K(z) = n^{-\beta\sigma_3/2} M \begin{pmatrix} 1 & 0 \\ \dfrac{e^{2n\phi}}{\pm H} & 1 \end{pmatrix} M^{-1} n^{\beta\sigma_3/2}. \tag{2.125}$$

For $z \in \Sigma^7_{K,\pm}$, we have

$$J_K(z) = n^{-\beta\sigma_3/2} M \begin{pmatrix} 1 & \dfrac{\pm\widetilde{H}\widetilde{E}}{e^{2n\widetilde{\phi}} E} \\ 0 & 1 \end{pmatrix} M^{-1} n^{\beta\sigma_3/2}. \tag{2.126}$$

For $z \in \Sigma^b_K$, we have

$$J_K(z) = \sqrt{\pi} n^{-\beta\sigma_3/2} H^{\sigma_3/2} m F^{\sigma_3/4} A(F) e^{n\phi\sigma_3} H^{-\sigma_3/2} M^{-1} n^{\beta\sigma_3/2}. \tag{2.127}$$

For $z \in \Sigma^a_K$, we have

$$J_K(z) = \sqrt{\pi} n^{-\beta\sigma_3/2} \widetilde{H}^{\sigma_3/2} \widetilde{m} \widetilde{F}^{-\sigma_3/4} \sigma_1 A(\widetilde{F}) \sigma_1 e^{n\widetilde{\phi}\sigma_3} \widetilde{H}^{-\sigma_3/2} M^{-1} n^{\beta\sigma_3/2}. \tag{2.128}$$

For $z \in \Sigma^0_K$, we have

$$J_K(z) = n^{-\beta\sigma_3/2} M D^{\sigma_3} M^{-1} n^{\beta\sigma_3/2}. \tag{2.129}$$

For $z \in \Sigma'_{K,\pm}$, we have

$$J_K(z) = n^{-\beta\sigma_3/2} M \begin{pmatrix} 1 & 0 \\ \dfrac{e^{2n\phi}}{\mp H D_+ D_-} & 1 \end{pmatrix} M^{-1} n^{\beta\sigma_3/2}. \tag{2.130}$$

Furthermore, the jump conditions of $K(z)$ on the positive real line are given as

$$J_K(x) = n^{-\beta\sigma_3/2} M \begin{pmatrix} 1 & 0 \\ \dfrac{e^{2n\widetilde{\phi}}}{\widetilde{H} D^2} & 1 \end{pmatrix} M^{-1} n^{\beta\sigma_3/2} \tag{2.131a}$$

for $0 < x < \varepsilon$, and

$$J_K(x) = n^{-\beta\sigma_3/2} M \begin{pmatrix} 1 & 0 \\ \dfrac{e^{2n\widetilde{\phi}}}{\widetilde{H}} & 1 \end{pmatrix} M^{-1} n^{\beta\sigma_3/2} \tag{2.131b}$$

for $\varepsilon < x < a - \varepsilon$*, and*

$$J_K(x) = n^{-\beta\sigma_3/2} M \begin{pmatrix} 1 & -\dfrac{H}{e^{2n\phi}} \\ 0 & 1 \end{pmatrix} M^{-1} n^{\beta\sigma_3/2} \tag{2.131c}$$

for $x > b + \varepsilon$*. On the contour* $\Sigma_K \setminus (\Sigma_K^a \cup \Sigma_K^b \cup \Sigma_K^0)$*, the* L^∞ *and* L^1 *norms of the difference* $J_K - I$ *are exponentially small as* $n \to \infty$*. On the contour* $\Sigma_K^a \cup \Sigma_K^b \cup \Sigma_K^0$*, we have*

$$\|J_K - I\|_{L^\infty(\Sigma_K^a \cup \Sigma_K^b \cup \Sigma_K^0)} = O(\frac{1}{n}), \qquad n \to \infty.$$

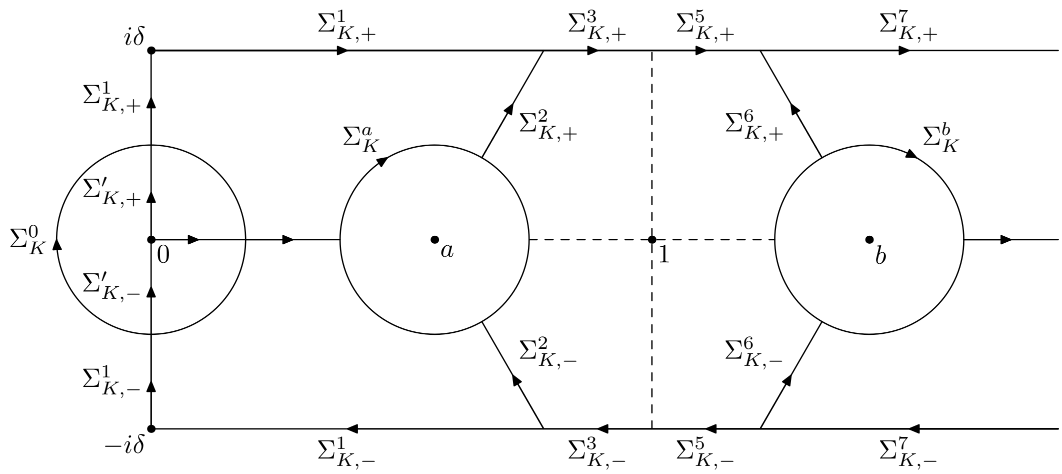

Figure 2.7 The contour Σ_K.

Proof. In Remark 2.18 we have shown that the function $F(z)$ is analytic in $U(b, \delta_0)$ and the function $\widetilde{F}(z)$ is analytic in $U(a, \delta_0)$. Since $0 < \varepsilon < \delta < \delta_0$, we obtain from (2.106) and (2.119) that the matrix-valued function $mF^{\sigma_3/4}$ is analytic in $U(b, \varepsilon)$, and from (2.104) and (2.120) that the matrix-valued function $\widetilde{m}\widetilde{F}^{-\sigma_3/4}$ is analytic in $U(a, \varepsilon)$. Therefore, applying (2.111) to (2.116) and (2.117) implies that the parametrix $T_{par}(z)$ possesses the same jump conditions as $T(z)$ in $U(a, \varepsilon) \cup U(b, \varepsilon)$; see (2.87) and (2.88) in Proposition 2.14. Thus, the function $K(z)$ defined in (2.121) is analytic in $U(a, \varepsilon) \cup U(b, \varepsilon)$. Moreover, applying (2.87), (2.92) and (2.114) to (2.121) implies that the function $K(z)$ can be analytically continued to the interval $(a+\varepsilon, b-\varepsilon)$. Therefore, the analyticity of $K(z)$ in $\mathbb{C}\setminus\Sigma_K$ is clear from the analyticity of $T(z)$ in $\mathbb{C}\setminus\Sigma_T$.

Since the function $M(z)$ is analytic in $\mathbb{C}\setminus[a,b]$, we obtain (2.122)-(2.126) from (2.83)-(2.86), (2.88), (2.114) and (2.121).

Since $T(z)$ has no jump on $\Sigma_K^a\cup\Sigma_K^b\cup\Sigma_K^0$, the formulas (2.127)-(2.129) follow immediately from the definition of $T_{par}(z)$ in (2.114)-(2.117), and from the definition of $K(z)$ in (2.121).

For $z\in\Sigma'_{K,\pm}$, applying (2.115) to (2.121) gives

$$J_K(z) = n^{-\beta\sigma_3/2} M D_-^{\sigma_3} T_-^{-1} T_+ D_+^{-\sigma_3} M^{-1} n^{\beta\sigma_3/2}.$$

Thus, formula (2.130) follows from (2.85) and (2.92).

Moreover, a combination of (2.87), (2.114), (2.115) and (2.121) yields (2.131).

From (2.100), (2.101) and (2.103), we obtain

$$|n^{-\beta\sigma_3/2} M(z) n^{\beta\sigma_3/2}| = O(1), \qquad n\to\infty. \tag{2.132}$$

By applying (2.47)-(2.52), (2.58), (2.97), (2.109) and (2.132) to (2.122)-(2.126) and (2.129)-(2.131), it follows that the norm $\|J_K - I\|_{L^\infty(\Sigma_K\setminus(\Sigma_K^a\cup\Sigma_K^b\cup\Sigma_K^0))}$ is exponentially small as $n\to\infty$.

To prove the norm $\|J_K - I\|_{L^1(\Sigma_K\setminus(\Sigma_K^a\cup\Sigma_K^b\cup\Sigma_K^0))}$ decays exponentially as $n\to\infty$, we only need to show the L^1 norm of the difference $J_K - I$ on the infinite contour $\Sigma_{K,\pm}^7\cup(b+\varepsilon,\infty)$ is exponentially small as $n\to\infty$. Firstly, since $\phi''(x)>0$ for $x>b$ by (2.37) and the fact that $ab=1$, we have

$$\phi(x) > \phi(b+\varepsilon) + (x-b-\varepsilon)\phi'(b+\varepsilon)$$

for any $x>b+\varepsilon$. Hence, we obtain

$$\|e^{-2n\phi}\|_{L^1(b+\varepsilon,\infty)} \le \frac{e^{-2n\phi(b+\varepsilon)}}{2n\phi'(b+\varepsilon)}.$$

Applying (2.97) and (2.132) to (2.131) implies that the norm $\|J_K - I\|_{L^1(b+\varepsilon,\infty)}$ is exponentially small as $n\to\infty$. Furthermore, we observe from (2.37), (2.50) and (2.51) that the L^1 norm of the function $e^{-2n\widetilde{\phi}}$ on the contour $\Sigma^7_{K,\pm}$ is also exponentially small as $n \to \infty$. Therefore, applying (2.58), (2.97) and (2.101) to (2.126) implies that the norm $\|J_K - I\|_{L^1(\Sigma^7_{K,\pm})}$ is exponentially small as $n \to \infty$. Thus, the exponential decay property of the norm $\|J_K - I\|_{L^1(\Sigma_K \setminus (\Sigma^a_K \cup \Sigma^b_K \cup \Sigma^0_K))}$ follows.

Now, we prove the last statement of the proposition. For $z \in \Sigma^b_K$, applying (2.107), (2.113) and (2.118) to (2.127) yields

$$J_K(z) - I = n^{-\beta\sigma_3/2} H(z)^{\sigma_3/2} m(z) O(\frac{1}{n}) m(z)^{-1} H(z)^{-\sigma_3/2} n^{\beta\sigma_3/2}, \qquad n \to \infty.$$

The estimate holds uniformly for $z \in \Sigma^b_K$. Thus, we obtain from (2.96)

$$\|J_K - I\|_{L^\infty(\Sigma^b_K)} = O(\frac{1}{n}), \qquad n \to \infty.$$

Similarly, a combination of (2.96), (2.105), (2.113), (2.118) and (2.128) gives

$$\|J_K - I\|_{L^\infty(\Sigma^a_K)} = O(\frac{1}{n}), \qquad n \to \infty.$$

Finally, by (2.109) we have $D(z) = 1 + O(1/n)$ uniformly for $z \in \Sigma^0_K$. Hence, it follows from (2.129) and (2.132) that

$$\|J_K - I\|_{L^\infty(\Sigma^0_K)} = O(\frac{1}{n}), \qquad n \to \infty.$$

This ends the proof of the proposition. □

Proposition 2.20. *The matrix-valued function $K(z)$ defined in (2.121) is the unique solution to the Riemann-Hilbert problem:*

(K1) $K(z)$ is analytic in $\mathbb{C} \setminus \Sigma_K$;

(K2) for $z \in \Sigma_K$, $K_+(z) = K_-(z) J_K(z)$, where $J_K(z)$ is given in Proposition 2.19;

(K3) for $z \in \mathbb{C} \setminus \Sigma_K$, $K(z) = I + O(|z|^{-1})$ as $z \to \infty$.

Furthermore, as $n \to \infty$, we have $K(z) = I + O(1/n)$ uniformly for $z \in \mathbb{C} \setminus \Sigma_K$.

Proof. The analyticity condition (K1) and the jump conditions (K2) have been shown in Proposition 2.19. The normalization condition (K3) is clear from the normalization conditions of the functions $T(z)$ and $M(z)$, and from the definition of the function $K(z)$. The uniqueness again follows from Liouville's theorem. Finally, as in [7, Theorem 7.10], we can obtain from Proposition 2.19 that $K(z) = I + O(1/n)$ as $n \to \infty$. The estimate is uniform for all $z \in \mathbb{C} \setminus \Sigma_K$. □

2.4 Uniform asymptotic formulas for the Meixner polynomials

Theorem 2.21. *For any $0 < c < 1$ and $1 \le \beta < 2$, let $\delta_0 > 0$ be a sufficiently small number depending only on the constants c and β; see Remark 2.9. Recall from (2.21) and (2.38) that $v(z) = -z \log c$ and $l/2 = \log \frac{b-a}{4} - 1$, where a and b are the Mhaskar-Rakhmanov-Saff numbers given in (2.31). The functions $g, \phi, \widetilde{\phi}$ and D are defined in (2.33), (2.37), (2.39) and (2.109), respectively. For any $0 < \varepsilon < \delta < \delta_0$, the large – n behavior of the monic Meixner polynomial $\pi_n(nz - \beta/2)$ is given below (see Figure 2.8).*

(i) For $z \in \Omega^4 \cup \Omega^\infty$, we have

$$\pi_n(nz - \beta/2) = n^n e^{ng(z)} \frac{z^{(1-\beta)/2} \left(\frac{\sqrt{z-a}+\sqrt{z-b}}{2}\right)^\beta}{(z-a)^{1/4}(z-b)^{1/4}} \left[1 + O(\frac{1}{n})\right]. \quad (2.133)$$

(ii) For $z \in \Omega^1_\pm$, we have

$$\begin{aligned}\pi_n(nz - \beta/2) = &-2(-n)^n e^{nv(z)/2+nl/2} \frac{z^{(1-\beta)/2} \left(\frac{\sqrt{b-z}+\sqrt{a-z}}{2}\right)^\beta}{(a-z)^{1/4}(b-z)^{1/4}} \\ &\times \Bigg\{ \sin(n\pi z - \beta\pi/2) e^{-n\widetilde{\phi}(z)} \left[1 + O(\frac{1}{n})\right] \\ &+ O(n^\beta e^{n \operatorname{Re} \phi(z)}) \Bigg\}. \end{aligned} \quad (2.134)$$

(iii) For $z \in \Omega_l^0$, we have

$$\pi_n(nz-\beta/2) = D(z) n^n e^{ng(z)} \frac{(-z)^{(1-\beta)/2}(\frac{\sqrt{b-z}+\sqrt{a-z}}{2})^\beta}{(b-z)^{1/4}(a-z)^{1/4}} \times \left[1+O(\frac{1}{n})\right]. \tag{2.135}$$

(iv) For $z \in \Omega_{r,\pm}^0$, we have

$$\pi_n(nz-\beta/2) = -2D(z)(-n)^n e^{nv(z)/2+nl/2} \frac{z^{(1-\beta)/2}(\frac{\sqrt{b-z}+\sqrt{a-z}}{2})^\beta}{(a-z)^{1/4}(b-z)^{1/4}} \times\left\{ \sin(n\pi z-\beta\pi/2) e^{-n\widetilde{\phi}(z)}\left[1+O(\frac{1}{n})\right] + O(n^\beta e^{n\operatorname{Re}\phi(z)})\right\}. \tag{2.136}$$

(v) Recall the definition of the functions $F(z)$ and $\widetilde{F}(z)$ in (2.118). For $z \in \Omega^a$, we have

$$\pi_n(nz-\beta/2) = (-n)^n \sqrt{\pi} e^{nv(z)/2+nl/2}\left\{\widetilde{\mathbf{A}}(z,n)\left[1+O(\frac{1}{n})\right] + \widetilde{\mathbf{B}}(z,n)\left[1+O(\frac{1}{n})\right]\right\}, \tag{2.137}$$

where

$$\widetilde{\mathbf{A}}(z,n) := \frac{(\frac{\sqrt{b-z}+\sqrt{a-z}}{2})^\beta + (\frac{\sqrt{b-z}-\sqrt{a-z}}{2})^\beta}{z^{(\beta-1)/2}(b-z)^{1/4}(a-z)^{1/4}\widetilde{F}(z)^{-1/4}} \times[\cos(n\pi z-\beta\pi/2)\operatorname{Ai}(\widetilde{F}(z)) - \sin(n\pi z-\beta\pi/2)\operatorname{Bi}(\widetilde{F}(z))],$$

and

$$\widetilde{\mathbf{B}}(z,n) := \frac{(\frac{\sqrt{b-z}+\sqrt{a-z}}{2})^\beta - (\frac{\sqrt{b-z}-\sqrt{a-z}}{2})^\beta}{z^{(\beta-1)/2}(b-z)^{1/4}(a-z)^{1/4}\widetilde{F}(z)^{1/4}} \times[\cos(n\pi z-\beta\pi/2)\operatorname{Ai}'(\widetilde{F}(z)) - \sin(n\pi z-\beta\pi/2)\operatorname{Bi}'(\widetilde{F}(z))].$$

(vi) For $z \in \Omega^b$, we have

$$\pi_n(nz-\beta/2) = n^n \sqrt{\pi} e^{nv(z)/2+nl/2}\left\{\mathbf{A}(z,n)\left[1+O(\frac{1}{n})\right] + \mathbf{B}(z,n)\left[1+O(\frac{1}{n})\right]\right\}, \tag{2.138}$$

where

$$\mathbf{A}(z,n) := \frac{(\frac{\sqrt{z-a}+\sqrt{z-b}}{2})^{\beta} + (\frac{\sqrt{z-a}-\sqrt{z-b}}{2})^{\beta}}{z^{(\beta-1)/2}(z-a)^{1/4}(z-b)^{1/4}F(z)^{-1/4}} \operatorname{Ai}(F(z)),$$

and

$$\mathbf{B}(z,n) := -\frac{(\frac{\sqrt{z-a}+\sqrt{z-b}}{2})^{\beta} - (\frac{\sqrt{z-a}-\sqrt{z-b}}{2})^{\beta}}{z^{(\beta-1)/2}(z-a)^{1/4}(z-b)^{1/4}F(z)^{1/4}} \operatorname{Ai}'(F(z)).$$

(vii) Let $z = \frac{b-a}{2}\cos u + \frac{b+a}{2} = -\frac{b-a}{2}\cos\widetilde{u} + \frac{b+a}{2}$. *We have*

$$\begin{aligned}\pi_n(nz-\beta/2) = 2(-n)^n e^{nv(z)/2+nl/2} &\frac{z^{(1-\beta)/2}(\frac{b-a}{4})^{\beta/2}}{(z-a)^{1/4}(b-z)^{1/4}} \\ &\times\Big\{\cos(n\pi z - \beta\pi/2 + \pi/4 + \beta\widetilde{u}/2 \mp in\widetilde{\phi}(z))\left[1+O(\frac{1}{n})\right] \\ &+O(n^{-1}e^{n|\operatorname{Re}\widetilde{\phi}(z)|+n\pi|\operatorname{Im} z|})\Big\}\end{aligned} \tag{2.139}$$

for $z \in \Omega^2_{\pm}$, *and*

$$\begin{aligned}\pi_n(nz-\beta/2) = 2n^n e^{nv(z)/2+nl/2} &\frac{z^{(1-\beta)/2}(\frac{b-a}{4})^{\beta/2}}{(z-a)^{1/4}(b-z)^{1/4}} \\ &\times\Big\{\cos(\pi/4 - \beta u/2 \mp in\phi(z))\left[1+O(\frac{1}{n})\right] \\ &+O(n^{-1}e^{n|\operatorname{Re}\phi(z)|})\Big\}\end{aligned} \tag{2.140}$$

for $z \in \Omega^3_{\pm}$. *In view of (2.39) and the fact that* $\widetilde{u} + u = \pi$, *the asymptotic formulas (2.139) and (2.140) are exactly the same.*

Proof. By applying (2.10), (2.22) and (2.58) to (2.13) we obtain

$$U(z) = R(z)\left[\prod_{j=0}^{n-1}(z-X_j)\right]^{\sigma_3}\begin{pmatrix} 1 & 0 \\ \dfrac{\mp Ee^{nv}}{W\widetilde{E}} & 1\end{pmatrix} \tag{2.141a}$$

for $\operatorname{Re} z \in (0,1)$ and $\operatorname{Im} z \in (0, \pm\delta)$, and

$$U(z) = R(z)\left[\prod_{j=0}^{n-1}(z-X_j)\right]^{\sigma_3}\begin{pmatrix} 1 & \dfrac{\mp W\widetilde{E}e^{-nv}}{Ee^{\mp 2i\pi(nz-\beta/2)}} \\ 0 & 1\end{pmatrix} \tag{2.141b}$$

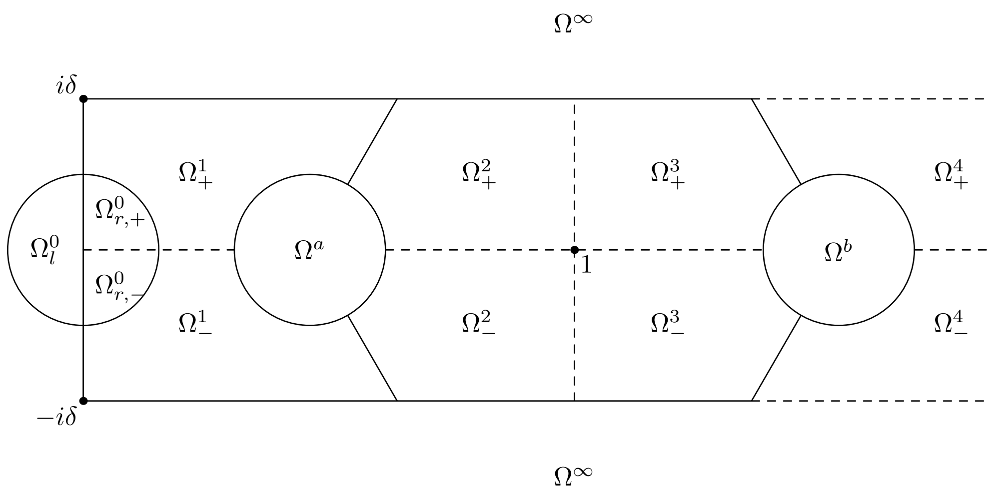

Figure 2.8 Regions of asymptotic approximations. A dashed line indicates that the asymptotic formulas on its two sides are the same.

for $\operatorname{Re} z \in (1, \infty)$ and $\operatorname{Im} z \in (0, \pm\delta)$, and

$$U(z) = R(z) \left[\prod_{j=0}^{n-1} (z - X_j) \right]^{\sigma_3} \tag{2.141c}$$

for $\operatorname{Re} z \notin [0, \infty)$ or $\operatorname{Im} z \notin [-\delta, \delta]$. It is easily seen from (2.53) and (2.60) that

$$\prod_{j=0}^{n-1} (z - X_j) = \left(\frac{z}{z-1} \right)^{\frac{1-\beta}{2}} E(z) e^{ng(z) - G(z)}. \tag{2.142}$$

For the sake of convenience, we put

$$\widetilde{U}(z) := e^{(-nl/2)\sigma_3} U(z) e^{(-nv(z)/2)\sigma_3}. \tag{2.143}$$

Thus, we have from (2.6) and (2.8) that

$$\widetilde{U}_{11}(z) = n^{-n} e^{-nv(z)/2 - nl/2} \pi_n(nz - \beta/2). \tag{2.144}$$

A combination of (2.40), (2.55), (2.62) and (2.141)-(2.143) yields

$$\widetilde{U}(z) = S(z) E^{\sigma_3} e^{-n\phi\sigma_3} \begin{pmatrix} 1 & 0 \\ \dfrac{\mp E}{H\widetilde{E}} & 1 \end{pmatrix} \left(\frac{z}{z-1} \right)^{\frac{1-\beta}{2}\sigma_3} \tag{2.145a}$$

for $\operatorname{Re} z \in (0,1)$ and $\operatorname{Im} z \in (0, \pm\delta)$, and

$$\widetilde{U}(z) = S(z) E^{\sigma_3} e^{-n\phi\sigma_3} \begin{pmatrix} 1 & \dfrac{\mp H\widetilde{E}}{E e^{\mp 2i\pi(nz-\beta/2)}} \\ 0 & 1 \end{pmatrix} \left(\frac{z}{z-1}\right)^{\frac{1-\beta}{2}\sigma_3} \tag{2.145b}$$

for $\operatorname{Re} z \in (1,\infty)$ and $\operatorname{Im} z \in (0, \pm\delta)$, and

$$\widetilde{U}(z) = S(z) E^{\sigma_3} e^{-n\phi\sigma_3} \left(\frac{z}{z-1}\right)^{\frac{1-\beta}{2}\sigma_3} \tag{2.145c}$$

for $\operatorname{Re} z \notin [0,\infty)$ or $\operatorname{Im} z \notin [-\delta, \delta]$.

For $z \in \Omega^4 \cup \Omega^\infty$, we apply (2.81), (2.114) and (2.121) to (2.145), and obtain

$$\begin{aligned} e^{-L\sigma_3/2} \widetilde{U} e^{L\sigma_3/2} = & (e^{-L\sigma_3/2} n^{\beta\sigma_3/2} K n^{-\beta\sigma_3/2} e^{L\sigma_3/2})(e^{-L\sigma_3/2} M e^{L\sigma_3/2}) \\ & \times e^{-n\phi\sigma_3} \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \left(\frac{z}{z-1}\right)^{\frac{1-\beta}{2}\sigma_3}; \end{aligned} \tag{2.146}$$

here and below, we denote by $*$ some irrelevant quantity which does not effect our final result. From (2.101) and Proposition 2.20, we have

$$e^{-L\sigma_3/2} n^{\beta\sigma_3/2} K n^{-\beta\sigma_3/2} e^{L\sigma_3/2} = I + O(\frac{1}{n}), \qquad n \to \infty. \tag{2.147}$$

Therefore, applying (2.100) and (2.103) to (2.146) yields

$$\widetilde{U}_{11} = e^{-n\phi} \frac{z^{\frac{1-\beta}{2}} (\frac{\sqrt{z-a}+\sqrt{z-b}}{2})^{\beta}}{(z-a)^{1/4}(z-b)^{1/4}} \left[1 + O(\frac{1}{n})\right].$$

Hence the asymptotic formula (2.133) follows from (2.40) and (2.144).

For $z \in \Omega^1_\pm$, we apply (2.81), (2.114) and (2.121) to (2.145). The result is

$$\begin{aligned} e^{-L\sigma_3/2} \widetilde{U} e^{L\sigma_3/2} = & (e^{-L\sigma_3/2} n^{\beta\sigma_3/2} K n^{-\beta\sigma_3/2} e^{L\sigma_3/2})(e^{-L\sigma_3/2} M e^{L\sigma_3/2}) e^{-n\phi\sigma_3} \\ & \times \begin{pmatrix} E/\widetilde{E} & 0 \\ \mp e^L/H & \widetilde{E}/E \end{pmatrix} \left(\frac{z}{z-1}\right)^{\frac{1-\beta}{2}\sigma_3}. \end{aligned} \tag{2.148}$$

Recall from (2.22) and (2.55) that $H(z) = [z/(z-1)]^{1-\beta} W(z)$ and

$$W(z) = \frac{2ni\pi\Gamma(nz+\beta/2)c^{-\beta/2}}{\Gamma(nz+1-\beta/2)}.$$

As $n \to \infty$, we observe by Stirling's formula that the function $H(z)^{-1}$ is uniformly bounded for $\operatorname{Re} z \geq 0$ and $z \neq 1$. From (2.101), it follows that $|e^L/H(z)| = O(n^\beta)$. Therefore, applying (2.100), (2.103) and (2.147) to (2.148) gives

$$\widetilde{U}_{11} = \frac{z^{\frac{1-\beta}{2}}(\frac{\sqrt{b-z}+\sqrt{a-z}}{2})^\beta}{(b-z)^{1/4}(a-z)^{1/4}}\Big\{(E/\widetilde{E})e^{-n\phi\mp i\pi(1-\beta)/2}\left[1+O(\frac{1}{n})\right] \\ +O(n^\beta e^{n\operatorname{Re}\phi})\Big\}. \tag{2.149}$$

Since

$$(E/\widetilde{E})e^{-n\phi\mp i\pi(1-\beta)/2} = -2(-1)^n e^{-n\widetilde{\phi}}\sin(n\pi z - \beta\pi/2)$$

by (2.39) and (2.58), the asymptotic formula (2.134) follows from (2.144) and (2.149).

For $z \in \Omega^0$, the proof of (2.135) and (2.136) is similar to that of (2.133) and (2.134). The only difference comes from the definition of the parametrix $T_{par}(z)$ in (2.114) and (2.115). We thus replace M by MD^{σ_3} in (2.146) and (2.148); consequently, the asymptotic formulas (2.135) and (2.136) are simply the formulas (2.133) and (2.134) multiplied by the function $D(z)$.

For $z \in \Omega^a$, we first consider the case $\arg \widetilde{F}(z) \in (\mp 2\pi/3, \mp\pi)$. In view of (2.120), this region is approximately the same as the region $\arg(z-a) \in (0, \pm\pi/3)$. Hence, we obtain from (2.81) and Remark 2.18 that

$$T(z) = S(z)\begin{pmatrix} \widetilde{E} & \mp\widetilde{H}\widetilde{E}/e^{2n\widetilde{\phi}} \\ 0 & 1/\widetilde{E} \end{pmatrix}.$$

Applying this and (2.121) to (2.145) gives

$$\widetilde{H}^{-\sigma_3/2}\widetilde{U}\widetilde{H}^{\sigma_3/2} = (\widetilde{H}^{-\sigma_3/2}n^{\beta\sigma_3/2}Kn^{-\beta\sigma_3/2}\widetilde{H}^{\sigma_3/2})(\widetilde{H}^{-\sigma_3/2}T_{par}\widetilde{H}^{\sigma_3/2}e^{-n\widetilde{\phi}\sigma_3}) \\ \times e^{n\widetilde{\phi}\sigma_3}\widetilde{H}^{-\sigma_3/2}\begin{pmatrix} 1/\widetilde{E} & \dfrac{\pm\widetilde{H}\widetilde{E}}{e^{2n\widetilde{\phi}}} \\ 0 & \widetilde{E} \end{pmatrix} e^{-n\phi\sigma_3}\begin{pmatrix} E & 0 \\ \dfrac{\mp 1}{H\widetilde{E}} & 1/E \end{pmatrix}\widetilde{H}^{\sigma_3/2} \\ \times\left(\frac{z}{z-1}\right)^{\frac{1-\beta}{2}\sigma_3}. \tag{2.150}$$

Coupling (2.110) and (2.117) yields

$$
\begin{aligned}
&\widetilde{H}^{-\sigma_3/2} T_{par} \widetilde{H}^{\sigma_3/2} e^{-n\widetilde{\phi}\sigma_3} \\
&= \sqrt{\pi}\widetilde{m}\widetilde{F}^{-\sigma_3/4} \begin{pmatrix} -i\omega^2 \operatorname{Ai}'(\omega\widetilde{F}) & -i\omega \operatorname{Ai}'(\omega^2\widetilde{F}) \\ -\omega \operatorname{Ai}(\omega\widetilde{F}) & -\omega^2 \operatorname{Ai}(\omega^2\widetilde{F}) \end{pmatrix} \\
&= \sqrt{\pi}\widetilde{m}\widetilde{F}^{-\sigma_3/4} \begin{pmatrix} \operatorname{Ai}'(\widetilde{F}) & -\operatorname{Bi}'(\widetilde{F}) \\ -i\operatorname{Ai}(\widetilde{F}) & i\operatorname{Bi}(\widetilde{F}) \end{pmatrix} \begin{pmatrix} i/2 & i/2 \\ -1/2 & 1/2 \end{pmatrix}
\end{aligned} \tag{2.151}
$$

for $\arg \widetilde{F}(z) \in (-2\pi/3, -\pi)$, and

$$
\begin{aligned}
&\widetilde{H}^{-\sigma_3/2} T_{par} \widetilde{H}^{\sigma_3/2} e^{-n\widetilde{\phi}\sigma_3} \\
&= \sqrt{\pi}\widetilde{m}\widetilde{F}^{-\sigma_3/4} \begin{pmatrix} i\omega \operatorname{Ai}'(\omega^2\widetilde{F}) & -i\omega^2 \operatorname{Ai}'(\omega\widetilde{F}) \\ \omega^2 \operatorname{Ai}(\omega^2\widetilde{F}) & -\omega \operatorname{Ai}(\omega\widetilde{F}) \end{pmatrix} \\
&= \sqrt{\pi}\widetilde{m}\widetilde{F}^{-\sigma_3/4} \begin{pmatrix} \operatorname{Ai}'(\widetilde{F}) & -\operatorname{Bi}'(\widetilde{F}) \\ -i\operatorname{Ai}(\widetilde{F}) & i\operatorname{Bi}(\widetilde{F}) \end{pmatrix} \begin{pmatrix} -i/2 & i/2 \\ -1/2 & -1/2 \end{pmatrix}
\end{aligned} \tag{2.152}
$$

for $\arg \widetilde{F}(z) \in (2\pi/3, \pi)$. Here, we have made use of the identities

$$
2\omega \operatorname{Ai}(\omega z) = -\operatorname{Ai}(z) + i\operatorname{Bi}(z), \qquad 2\omega^2 \operatorname{Ai}(\omega^2 z) = -\operatorname{Ai}(z) - i\operatorname{Bi}(z). \tag{2.153}
$$

A combination of (2.39), (2.58) and (2.59) implies

$$
\begin{aligned}
&e^{n\widetilde{\phi}\sigma_3}\widetilde{H}^{-\sigma_3/2} \begin{pmatrix} 1/\widetilde{E} & \dfrac{\pm\widetilde{H}\widetilde{E}}{e^{2n\widetilde{\phi}}} \\ 0 & \widetilde{E} \end{pmatrix} e^{-n\phi\sigma_3} \begin{pmatrix} E & 0 \\ \dfrac{\mp 1}{\widetilde{E}H} & 1/E \end{pmatrix} \widetilde{H}^{\sigma_3/2} \left(\frac{z}{z-1}\right)^{\frac{1-\beta}{2}\sigma_3} \\
&= (-1)^n \begin{pmatrix} \mp i e^{\mp i\pi(nz-\beta/2)} & i e^{\pm i\pi(nz-\beta/2)}\widetilde{E}/E \\ -i e^{\pm i\pi(nz-\beta/2)} & \pm i e^{\pm i\pi(nz-\beta/2)}\widetilde{E}/E \end{pmatrix} \left(\frac{z}{1-z}\right)^{\frac{1-\beta}{2}\sigma_3} \\
&= (-1)^n \begin{pmatrix} \mp i & -1 \\ -i & \pm 1 \end{pmatrix} \begin{pmatrix} \cos(n\pi z - \beta\pi/2) & * \\ \sin(n\pi z - \beta\pi/2) & * \end{pmatrix} \left(\frac{z}{1-z}\right)^{\frac{1-\beta}{2}\sigma_3},
\end{aligned} \tag{2.154}
$$

where the $*$ stands for some irrelevant quantities. Applying (2.151)-(2.154) to (2.150) gives

$$\begin{aligned}\widetilde{H}^{-\sigma_3/2}\widetilde{U}\widetilde{H}^{\sigma_3/2} = {} & (\widetilde{H}^{-\sigma_3/2}n^{\beta\sigma_3/2}Kn^{-\beta\sigma_3/2}\widetilde{H}^{\sigma_3/2})(\sqrt{\pi}\widetilde{m}\widetilde{F}^{-\sigma_3/4})(-1)^n \\ & \times\begin{pmatrix} \cos(n\pi z-\beta\pi/2)\,\mathrm{Ai}'(\widetilde{F})-\sin(n\pi z-\beta\pi/2)\,\mathrm{Bi}'(\widetilde{F}) & * \\ -i\cos(n\pi z-\beta\pi/2)\,\mathrm{Ai}(\widetilde{F})+i\sin(n\pi z-\beta\pi/2)\,\mathrm{Bi}(\widetilde{F}) & * \end{pmatrix} \\ & \times\left(\frac{z}{1-z}\right)^{\frac{1-\beta}{2}\sigma_3}. \end{aligned} \tag{2.155}$$

From (2.96) and Proposition 2.20, we have

$$\widetilde{H}^{-\sigma_3/2}n^{\beta\sigma_3/2}Kn^{-\beta\sigma_3/2}\widetilde{H}^{\sigma_3/2} = I + O(1/n), \qquad n\to\infty. \tag{2.156}$$

Coupling (2.155) and (2.156) yields

$$\begin{aligned}\widetilde{U}_{11}(z) = {} & (-1)^n\sqrt{\pi}\left(\frac{z}{1-z}\right)^{\frac{1-\beta}{2}} \\ & \times\left\{\widetilde{m}_{11}\widetilde{F}^{-1/4}r_{11}\left[1+O(\frac{1}{n})\right]+\widetilde{m}_{12}\widetilde{F}^{1/4}r_{21}\left[1+O(\frac{1}{n})\right]\right\},\end{aligned}$$

where

$$r_{11} := \cos(n\pi z-\beta\pi/2)\,\mathrm{Ai}'(\widetilde{F})-\sin(n\pi z-\beta\pi/2)\,\mathrm{Bi}'(\widetilde{F}),$$

and

$$r_{21} := -i\cos(n\pi z-\beta\pi/2)\,\mathrm{Ai}(\widetilde{F})+i\sin(n\pi z-\beta\pi/2)\,\mathrm{Bi}(\widetilde{F}).$$

Thus, formula (2.137) follows from (2.104) and (2.144).

Now, we consider the case $\arg\widetilde{F}(z)\in(0,\mp2\pi/3)$. In view of Remark 2.18, we obtain from (2.81) that $T(z) = S(z)\widetilde{E}(z)^{\sigma_3}$. Applying this and (2.121) to (2.145) gives

$$\begin{aligned}\widetilde{H}^{-\sigma_3/2}\widetilde{U}\widetilde{H}^{\sigma_3/2} = {} & (\widetilde{H}^{-\sigma_3/2}n^{\beta\sigma_3/2}Kn^{-\beta\sigma_3/2}\widetilde{H}^{\sigma_3/2})(\widetilde{H}^{-\sigma_3/2}T_{par}\widetilde{H}^{\sigma_3/2}e^{-n\widetilde{\phi}\sigma_3}) \\ & \times e^{n\widetilde{\phi}\sigma_3}\widetilde{H}^{-\sigma_3/2}e^{-n\phi\sigma_3}\begin{pmatrix} E/\widetilde{E} & 0 \\ \mp1/H & \widetilde{E}/E\end{pmatrix}\widetilde{H}^{\sigma_3/2} \\ & \times\left(\frac{z}{z-1}\right)^{\frac{1-\beta}{2}\sigma_3}. \end{aligned} \tag{2.157}$$

A combination of (2.110), (2.117) and (2.153) yields

$$\widetilde{H}^{-\sigma_3/2}T_{par}\widetilde{H}^{\sigma_3/2}e^{-n\widetilde{\phi}\sigma_3} = \sqrt{\pi}\widetilde{m}\widetilde{F}^{-\sigma_3/4}\begin{pmatrix} \mathrm{Ai}'(\widetilde{F}) & -\mathrm{Bi}'(\widetilde{F}) \\ -i\,\mathrm{Ai}(\widetilde{F}) & i\,\mathrm{Bi}(\widetilde{F}) \end{pmatrix}\begin{pmatrix} \pm i/2 & i \\ -1/2 & 0 \end{pmatrix}. \tag{2.158}$$

From (2.39), (2.58) and (2.59) we have

$$\begin{aligned} &e^{n\widetilde{\phi}\sigma_3}\widetilde{H}^{-\sigma_3/2}e^{-n\phi\sigma_3}\begin{pmatrix} E/\widetilde{E} & 0 \\ \mp 1/H & \widetilde{E}/E \end{pmatrix}\widetilde{H}^{\sigma_3/2}\left(\frac{z}{z-1}\right)^{\frac{1-\beta}{2}\sigma_3} \\ &= (-1)^n\begin{pmatrix} -2\sin(n\pi z-\beta\pi/2) & 0 \\ -ie^{\pm i\pi(nz-\beta/2)} & \pm ie^{\pm i\pi(nz-\beta/2)}\widetilde{E}/E \end{pmatrix}\left(\frac{z}{1-z}\right)^{\frac{1-\beta}{2}\sigma_3} \\ &= (-1)^n\begin{pmatrix} 0 & -2 \\ -i & \pm 1 \end{pmatrix}\begin{pmatrix} \cos(n\pi z-\beta\pi/2) & * \\ \sin(n\pi z-\beta\pi/2) & 0 \end{pmatrix}\left(\frac{z}{1-z}\right)^{\frac{1-\beta}{2}\sigma_3}, \end{aligned} \tag{2.159}$$

where the $*$ again stands for some irrelevant quantity. Applying (2.158) and (2.159) to (2.157), we again obtain (2.155). A combination of (2.104), (2.144), (2.155) and (2.156) yields (2.137).

For $z \in \Omega^b$, we only consider the case $\arg F(z) \in (\pm 2\pi/3, \pm\pi)$ here. The case $\arg F(z) \in (0, \pm 2\pi/3)$ is much simpler and we omit the details. On account of Remark 2.18, we obtain from (2.81) that

$$T(z) = S(z)\begin{pmatrix} E & 0 \\ e^{2n\phi}/(\pm HE) & 1/E \end{pmatrix}.$$

Applying this and (2.121) to (2.145), we have

$$\begin{aligned} H^{-\sigma_3/2}\widetilde{U}H^{\sigma_3/2} = {} & (H^{-\sigma_3/2}n^{\beta\sigma_3/2}Kn^{-\beta\sigma_3/2}H^{\sigma_3/2})(H^{-\sigma_3/2}T_{par}H^{\sigma_3/2}e^{-n\phi\sigma_3}) \\ & \times e^{n\phi\sigma_3}H^{-\sigma_3/2}\begin{pmatrix} 1/E & 0 \\ \dfrac{e^{2n\phi}}{\mp HE} & E \end{pmatrix}e^{-n\phi\sigma_3}\begin{pmatrix} E & \dfrac{\mp H\widetilde{E}}{e^{\mp 2i\pi(nz-\beta/2)}} \\ 0 & 1/E \end{pmatrix}H^{\sigma_3/2} \\ & \times\left(\frac{z}{z-1}\right)^{\frac{1-\beta}{2}\sigma_3} \end{aligned}$$

A simple calculation gives

$$H^{-\sigma_3/2}\widetilde{U}H^{\sigma_3/2} = (H^{-\sigma_3/2}n^{\beta\sigma_3/2}Kn^{-\beta\sigma_3/2}H^{\sigma_3/2})(H^{-\sigma_3/2}T_{par}H^{\sigma_3/2}e^{-n\phi\sigma_3}) \times \begin{pmatrix} 1 & * \\ \mp 1 & * \end{pmatrix} \left(\frac{z}{z-1}\right)^{\frac{1-\beta}{2}\sigma_3}, \tag{2.160}$$

where the $*$ stands for some irrelevant quantity. Coupling (2.110) and (2.116) yields

$$\begin{aligned} &H^{-\sigma_3/2}T_{par}H^{\sigma_3/2}e^{-n\phi\sigma_3} \\ &= \sqrt{\pi}mF^{\sigma_3/4}\begin{pmatrix} -\omega\,\mathrm{Ai}(\omega F) & \omega^2\,\mathrm{Ai}(\omega^2 F) \\ -i\omega^2\,\mathrm{Ai}'(\omega F) & i\omega\,\mathrm{Ai}'(\omega^2 F) \end{pmatrix} \\ &= \sqrt{\pi}mF^{\sigma_3/4}\begin{pmatrix} \mathrm{Ai}(F) & \mathrm{Bi}(F) \\ i\,\mathrm{Ai}'(F) & i\,\mathrm{Bi}'(F) \end{pmatrix}\begin{pmatrix} 1/2 & -1/2 \\ -i/2 & -i/2 \end{pmatrix} \end{aligned} \tag{2.161}$$

for $\arg F(z) \in (2\pi/3, \pi)$, and

$$\begin{aligned} &H^{-\sigma_3/2}T_{par}H^{\sigma_3/2}e^{-n\phi\sigma_3} \\ &= \sqrt{\pi}mF^{\sigma_3/4}\begin{pmatrix} -\omega^2\,\mathrm{Ai}(\omega^2 F) & -\omega\,\mathrm{Ai}(\omega F) \\ -i\omega\,\mathrm{Ai}'(\omega^2 F) & -i\omega^2\,\mathrm{Ai}'(\omega F) \end{pmatrix} \\ &= \sqrt{\pi}mF^{\sigma_3/4}\begin{pmatrix} \mathrm{Ai}(F) & \mathrm{Bi}(F) \\ i\,\mathrm{Ai}'(F) & i\,\mathrm{Bi}'(F) \end{pmatrix}\begin{pmatrix} 1/2 & 1/2 \\ i/2 & -i/2 \end{pmatrix} \end{aligned} \tag{2.162}$$

for $\arg F(z) \in (-2\pi/3, -\pi)$. Here, we have made use of (2.153). Moreover, from (2.96) and Proposition 2.20, we obtain

$$H^{-\sigma_3/2}n^{\beta\sigma_3/2}Kn^{-\beta\sigma_3/2}H^{\sigma_3/2} = I + O(1/n), \qquad n \to \infty. \tag{2.163}$$

A combination of (2.160)-(2.163) gives

$$\widetilde{U}_{11}(z) = \sqrt{\pi}\left(\frac{z}{z-1}\right)^{\frac{1-\beta}{2}} \times \left\{ m_{11}F^{1/4}\,\mathrm{Ai}(F)\left[1 + O(\frac{1}{n})\right] + im_{12}F^{-1/4}\,\mathrm{Ai}'(F)\left[1 + O(\frac{1}{n})\right] \right\}.$$

Thus, formula (2.138) follows from (2.106) and (2.144).

For $z \in \Omega_\pm^2$, similar to the proof of (2.137) in the case when $\arg \widetilde{F}(z)$ belongs to $(\mp 2\pi/3, \mp\pi)$, the equality (2.150) follows from (2.81), (2.121) and (2.145). Also a combination of (2.39), (2.58) and (2.59) gives (2.154). Set $z = -\frac{b-a}{2}\cos\widetilde{u} + \frac{b+a}{2}$, and we have $\sqrt{b-z} \pm i\sqrt{z-a} = \sqrt{b-a}e^{\pm i\widetilde{u}/2}$. Since $\widetilde{H}(z) = (1-z)^{\beta-1}e^{-V(z)}$ by (2.94), and $|\widetilde{G}(z)| + |V(z)+L| = O(1/n)$ by (2.95), (2.100) and (2.101), it follows from (2.103) and (2.114) that

$$\begin{aligned}\widetilde{H}^{-\sigma_3/2}T_{par}\widetilde{H}^{\sigma_3/2}e^{-n\widetilde{\phi}\sigma_3} = & \frac{(1-z)^{\frac{1-\beta}{2}\sigma_3}(\frac{b-a}{4})^{\beta/2}}{(b-z)^{1/4}(z-a)^{1/4}} \\ & \times \left[\begin{pmatrix} e^{\mp i\beta\widetilde{u}/2 \pm i\pi/4} & ie^{\pm i\beta\widetilde{u}/2\pm i\pi/4} \\ -ie^{\pm i\beta\widetilde{u}/2\pm i\pi/4} & e^{\mp i\beta\widetilde{u}/2\pm i\pi/4}\end{pmatrix} + O(\frac{1}{n})\right] \\ & \times e^{-n\widetilde{\phi}\sigma_3}. \end{aligned} \tag{2.164}$$

Since

$$\begin{aligned} & \begin{pmatrix} e^{\mp i\widetilde{u}/2\pm i\pi/4} & ie^{\pm i\widetilde{u}/2\pm i\pi/4} \\ -ie^{\pm i\widetilde{u}/2\pm i\pi/4} & e^{\mp i\widetilde{u}/2\pm i\pi/4}\end{pmatrix} e^{-n\widetilde{\phi}\sigma_3} \\ = & \begin{pmatrix} 2\cos(\pi/4 + \beta\widetilde{u}/2 \mp in\widetilde{\phi}) & -2\sin(\pi/4+\beta\widetilde{u}/2\mp in\widetilde{\phi}) \\ O(e^{n|\operatorname{Re}\widetilde{\phi}|}) & O(e^{n|\operatorname{Re}\widetilde{\phi}|})\end{pmatrix} \begin{pmatrix} \pm i/2 & i/2 \\ -1/2 & \pm 1/2\end{pmatrix}, \end{aligned}$$

it follows from (2.150), (2.154) and (2.164) that

$$\begin{aligned}\widetilde{H}^{-\sigma_3/2}\widetilde{U}\widetilde{H}^{\sigma_3/2} = & \left(\widetilde{H}^{-\sigma_3/2}n^{\beta\sigma_3/2}Kn^{-\beta\sigma_3/2}\widetilde{H}^{\sigma_3/2}\right)\frac{(1-z)^{\frac{1-\beta}{2}\sigma_3}(\frac{b-a}{4})^{\beta/2}}{(b-z)^{1/4}(z-a)^{1/4}} \\ & \times \begin{pmatrix} \widetilde{r}_{11} & * \\ O(e^{n|\operatorname{Re}\widetilde{\phi}|+n\pi|\operatorname{Im} z|}) & * \end{pmatrix} \left(\frac{z}{1-z}\right)^{\frac{1-\beta}{2}\sigma_3}, \end{aligned} \tag{2.165}$$

where the $*$ stands for some irrelevant quantities, and

$$\widetilde{r}_{11} = \cos(n\pi z - \beta\pi/2 + \pi/4 + \beta\widetilde{u}/2 \mp in\widetilde{\phi})\left[1 + O(\frac{1}{n})\right] + O(n^{-1}e^{n|\operatorname{Re}\widetilde{\phi}|+n\pi|\operatorname{Im} z|}).$$

For $z \in \Omega_\pm^2$, we have from (2.94) and (2.95) that

$$|n^{-\beta}\widetilde{H}(z)(1-z)^{1-\beta}| + |(1-z)^{\beta-1}\widetilde{H}(z)^{-1}n^\beta| = O(1)$$

as $n \to \infty$. In view of $K(z) = I + O(1/n)$ by Proposition 2.20, we obtain from (2.165)

$$\widetilde{U}_{11}(z) = \frac{2(-1)^n z^{\frac{1-\beta}{2}} (\frac{b-a}{4})^{\beta/2} \widetilde{r}_{11}}{(b-z)^{1/4}(z-a)^{1/4}}.$$

Coupling this with (2.144) yields (2.139).

For $z \in \Omega^3_{\pm}$, similar to the proof of (2.138) in the case $\arg F(z) \in (\pm 2\pi/3, \pm\pi)$, the equality (2.160) follows from (2.81), (2.121) and (2.145). Set $z = \frac{b-a}{2}\cos u + \frac{b+a}{2}$, and we have $\sqrt{z-a} \pm i\sqrt{b-z} = \sqrt{b-a}e^{\pm iu/2}$. Since $H(z) = (z-1)^{\beta-1}e^{-V(z)}$ by (2.94), and $|\widetilde{G}(z)| + |V(z)+L| = O(1/n)$ by (2.95), (2.100) and (2.101), it can be shown from (2.103) and (2.114) that

$$\begin{aligned} H^{-\sigma_3/2} T_{par} H^{\sigma_3/2} e^{-n\phi\sigma_3} = & \frac{(z-1)^{\frac{1-\beta}{2}\sigma_3} (\frac{b-a}{4})^{\beta/2}}{(b-z)^{1/4}(z-a)^{1/4}} \\ & \times \left[\begin{pmatrix} e^{\pm i\beta u/2 \mp i\pi/4} & -ie^{\mp i\beta u/2 \mp i\pi/4} \\ ie^{\mp i\beta u/2 \mp i\pi/4} & e^{\pm i\beta u/2 \mp i\pi/4} \end{pmatrix} + O(\frac{1}{n}) \right] \\ & \times e^{-n\phi\sigma_3}. \end{aligned} \tag{2.166}$$

Since

$$\begin{aligned} & \begin{pmatrix} e^{\pm i\beta u/2 \mp i\pi/4} & -ie^{\mp i\beta u/2 \mp i\pi/4} \\ ie^{\mp i\beta u/2 \mp i\pi/4} & e^{\pm i\beta u/2 \mp i\pi/4} \end{pmatrix} e^{-n\phi\sigma_3} \\ = & \begin{pmatrix} 2\cos(\pi/4 - \beta u/2 \mp in\phi) & -ie^{\mp i\beta u \mp i\pi/4 + n\phi} \\ O(e^{n|\operatorname{Re}\phi|}) & O(e^{n|\operatorname{Re}\phi|}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix}, \end{aligned}$$

applying (2.166) to (2.160) gives

$$\begin{aligned} H^{-\sigma_3/2} \widetilde{U} H^{\sigma_3/2} = & \left(H^{-\sigma_3/2} n^{\beta\sigma_3/2} K n^{-\beta\sigma_3/2} H^{\sigma_3/2} \right) \frac{(z-1)^{\frac{1-\beta}{2}\sigma_3} (\frac{b-a}{4})^{\beta/2}}{(b-z)^{1/4}(z-a)^{1/4}} \\ & \times \begin{pmatrix} r_{11} & * \\ O(e^{n|\operatorname{Re}\phi|}) & * \end{pmatrix} \left(\frac{z}{z-1} \right)^{\frac{1-\beta}{2}\sigma_3}, \end{aligned} \tag{2.167}$$

where the $*$ stands for some irrelevant quantities, and

$$r_{11} = \cos(\pi/4 - \beta u/2 \mp in\phi) \left[1 + O(\frac{1}{n}) \right] + O(n^{-1}e^{n|\operatorname{Re}\phi|}).$$

For $z \in \Omega_\pm^3$, we have from (2.94) and (2.95) that

$$|n^{-\beta}H(z)(z-1)^{1-\beta}| + |(z-1)^{\beta-1}H(z)^{-1}n^{\beta}| = O(1)$$

as $n \to \infty$. In view of $K(z) = I + O(1/n)$ by Proposition 2.20, we obtain from (2.167)

$$\widetilde{U}_{11}(z) = \frac{2z^{\frac{1-\beta}{2}}(\frac{b-a}{4})^{\beta/2}r_{11}}{(b-z)^{1/4}(z-a)^{1/4}}.$$

Coupling this with (2.144) yields (2.140). Moreover, since $z = \frac{b-a}{2}\cos u + \frac{b+a}{2} = -\frac{b-a}{2}\cos\widetilde{u} + \frac{b+a}{2}$, we have $\widetilde{u} + u = \pi$. In view of (2.39), the two asymptotic formulas (2.139) and (2.140) are exactly the same. $\square$

Chapter 3

Asymptotics of Some q-Orthogonal Polynomials

3.1 Discrete analogues of Laplace's approximation

In order to give applications to q-orthogonal polynomials, we need consider the sum

$$I_n(z|q) = \sum_{k=0}^{n} f_n(k) q^{g_n(k)} z^k, \tag{3.1}$$

where $q \in (0,1)$, f_n and g_n are real-valued functions defined on $\mathbb{N}$, and z is a complex variable. As we shall see, the large $-\, n$ behavior of $I_n(z|q)$ involves the q-Theta function (1.9)

$$\Theta_q(z) := \sum_{k=-\infty}^{\infty} q^{k^2} z^k.$$

Theorem 3.1. *Assume that the following conditions hold:*

(i) there is a number $l \in (0,1)$ such that $\lim_{n\to\infty} f_n(\lfloor nl \rfloor) = 1$ and $\lim_{n\to\infty} g_n(\lfloor nl \rfloor) = 0$;

(ii) there exists a constant $M > 0$ such that $|f_n(k)| \le M$ for $0 \le k \le n$;

(iii) for any $0 < \delta < l$, there exist $A_\delta > 0$ and $N(\delta) \in \mathbb{N}$ such that $g_n(k) \ge n^2 A_\delta$ for all $k \in [0, n(l-\delta)] \cup [n(l+\delta), n]$ and $n > N(\delta)$;

(iv) for any small $\varepsilon > 0$, there exist $\delta(\varepsilon) > 0$ and $N(\varepsilon) \in \mathbb{N}$ such that $|f_n(k) - 1| < \varepsilon$ and $|g_n(k) - b_n(k - \lfloor nl \rfloor) - c_0(k - \lfloor nl \rfloor)^2| < \varepsilon(k - \lfloor nl \rfloor)^2$ for $n(l - \delta(\varepsilon)) \le$

$k \le n(l+\delta(\varepsilon))$ *and* $n > N(\varepsilon)$, *where* $\sup_n |b_n| \le L$.

Then, we have

$$I_n(z|q) = z^{\lfloor nl \rfloor}[\Theta_{\widetilde{q}}(w_n) + o(1)] \qquad as \quad n \to \infty, \tag{3.2}$$

for all $z \in T_R := \{z \in \mathbb{C} : R^{-1} \le |z| \le R\}$, *where* $\widetilde{q} = q^{c_0}$ *and* $w_n = q^{b_n} z$.

Remark 3.2. *Condition (i) in Theorem 3.1 can always be satisfied, if we consider, instead of* $I_n(z|q)$, *the sum*

$$\widetilde{I}_n(z|q) = \frac{1}{f_n(\lfloor nl \rfloor)} q^{-g_n(\lfloor nl \rfloor)} I_n(z|q) = \sum_{k=0}^{n} \frac{f_n(k)}{f_n(\lfloor nl \rfloor)} q^{g_n(k) - g_n(\lfloor nl \rfloor)}.$$

Condition (iv) in the theorem is the discrete analogue of the conditions that f_n *is continuous and* g_n *is twice continuously differentiable at* $k = \lfloor nl \rfloor$ *with* $g_n'(\lfloor nl \rfloor) = b_n$ *and* $g_n''(\lfloor nl \rfloor) = 2c_0$.

Before proving Theorem 3.1, let us first establish the following stronger result.

Theorem 3.3. *Assume that the conditions (i), (ii) and (iii) in Theorem 3.1 hold. If condition (iv) in that theorem is strengthened to*

(iv′) for any small $\delta > 0$, *there exist a function* $\eta_n(\delta)$ *with* $\lim_{n\to\infty} \eta_n(\delta) = 0$ *and a positive integer* $N(\delta)$ *such that* $|f_n(k) - 1| \le \eta_n(\delta)$ *and* $|g_n(k) - b_n(k - \lfloor nl \rfloor) - c_0(k - \lfloor nl \rfloor)^2| \le \eta_n(\delta)(k - \lfloor nl \rfloor)^2$ *for all* k *in* $n(l-\delta) \le k \le n(l+\delta)$ *and all* $n > N(\delta)$,

then the error $r_n := z^{-\lfloor nl \rfloor} I_n(z|q) - \Theta_{\widetilde{q}}(w_n)$ *in the approximation (3.2) satisfies*

$$|r_n| \le C(\eta_n(\delta) + q^{n^2 A_\delta (1-\delta)} + q^{c_0 n^2 \delta^2 (1-\delta)}) \tag{3.3}$$

for sufficiently large n, *where* C *is a constant depending on* q, M, R, L, *and* c_0. *Furthermore, the estimate is uniform for* z *in the annulus* T_R *given in Theorem 3.1.*

Proof. Clearly,

$$r_n = \sum_{k=-\lfloor nl \rfloor}^{n-\lfloor nl \rfloor} f_n(k+\lfloor nl \rfloor) q^{g_n(k+\lfloor nl \rfloor)} z^k - \sum_{k=-\infty}^{\infty} q^{k^2 c_0 + k b_n} z^k.$$

We write the first sum as

$$\sum_{k=-\lfloor nl \rfloor}^{-\lfloor n\delta \rfloor - 1} + \sum_{k=-\lfloor n\delta \rfloor}^{\lfloor n\delta \rfloor} + \sum_{k=\lfloor n\delta \rfloor + 1}^{n-\lfloor nl \rfloor},$$

and the second sum as

$$\sum_{k=-\infty}^{-\lfloor n\delta \rfloor - 1} + \sum_{k=-\lfloor n\delta \rfloor}^{\lfloor n\delta \rfloor} + \sum_{k=\lfloor n\delta \rfloor + 1}^{\infty}.$$

Thus,

$$r_n = I_1 + I_2 + I_3 + I_4 + I_5 + I_6,$$

where

$$\begin{aligned}
I_1 &= \sum_{k=\lfloor n\delta \rfloor + 1}^{n-\lfloor nl \rfloor} f_n(k+\lfloor nl \rfloor) q^{g_n(k+\lfloor nl \rfloor)} z^k, \\
I_2 &= -\sum_{k=\lfloor n\delta \rfloor + 1}^{\infty} q^{k^2 c_0 + k b_n} z^k, \\
I_3 &= \sum_{k=-\lfloor nl \rfloor}^{-\lfloor n\delta \rfloor - 1} f_n(k+\lfloor nl \rfloor) q^{g_n(k+\lfloor nl \rfloor)} z^k, \\
I_4 &= -\sum_{k=-\infty}^{-\lfloor n\delta \rfloor - 1} q^{k^2 c_0 + k b_n} z^k, \\
I_5 &= \sum_{k=-\lfloor n\delta \rfloor}^{\lfloor n\delta \rfloor} f_n(k+\lfloor nl \rfloor) [q^{g_n(k+\lfloor nl \rfloor)} - q^{k^2 c_0 + k b_n}] z^k,
\end{aligned}$$

and

$$I_6 = -\sum_{k=-\lfloor n\delta \rfloor}^{\lfloor n\delta \rfloor} [f_n(k+\lfloor nl \rfloor) - 1] q^{k^2 c_0 + k b_n} z^k.$$

For sufficiently large n, we have

$$|I_1| \le \sum_{k=\lfloor n\delta \rfloor + 1}^{n-\lfloor nl \rfloor} M q^{n^2 A_\delta} R^k \le n M q^{n^2 A_\delta} R^n \le q^{n^2 A_\delta (1-\delta)},$$

and

$$
\begin{aligned}
|I_2| &\le \sum_{m=0}^{\infty} q^{m^2c_0+(\lfloor n\delta\rfloor+1)^2c_0-(m+\lfloor n\delta\rfloor+1)L}R^{m+\lfloor n\delta\rfloor+1} \\
&\le q^{(\lfloor n\delta\rfloor+1)^2c_0-(\lfloor n\delta\rfloor+1)L}R^{\lfloor n\delta\rfloor+1}\Theta_{q^{c_0}}(q^{-L}R) \\
&\le q^{c_0n^2\delta^2(1-\delta)},
\end{aligned}
$$

since $|b_n| \le L$. Similarly, we obtain

$$
|I_3| \le \sum_{k=-\lfloor nl\rfloor}^{-\lfloor n\delta\rfloor-1} Mq^{n^2A_\delta}R^{-k} \le nMq^{n^2A_\delta}R^n \le q^{n^2A_\delta(1-\delta)},
$$

and

$$
\begin{aligned}
|I_4| &\le \sum_{m=-\infty}^{0} q^{m^2c_0+(\lfloor n\delta\rfloor+1)^2c_0+(m-\lfloor n\delta\rfloor-1)L}R^{-m+\lfloor n\delta\rfloor+1} \\
&\le q^{(\lfloor n\delta\rfloor+1)^2c_0-(\lfloor n\delta\rfloor+1)L}R^{\lfloor n\delta\rfloor+1}\Theta_{q^{c_0}}(q^{-L}R) \\
&\le q^{c_0n^2\delta^2(1-\delta)}
\end{aligned}
$$

for large enough n.

We next estimate I_5 and I_6. It is evident that

$$
I_5 = \sum_{k=-\lfloor n\delta\rfloor}^{\lfloor n\delta\rfloor} f_n(k+\lfloor nl\rfloor)[q^{g_n(k+\lfloor nl\rfloor)-k^2c_0-kb_n}-1]q^{k^2c_0+kb_n}z^k.
$$

By the mean-value theorem, we have

$$
|I_5| \le M|\ln q|\eta_n(\delta)\sum_{k=-\lfloor n\delta\rfloor}^{\lfloor n\delta\rfloor} k^2q^{-\eta_n(\delta)k^2+k^2c_0+kb_n}|z|^k,
$$

where we have made use of condition (iv$'$). Since $e^{|k|} \ge \frac{1}{2}k^2$ and $\eta_n(\delta) \to 0$ as $n \to \infty$, the last inequality gives

$$
|I_5| \le 4M|\ln q|\eta_n(\delta)\Theta_{q^{c_0/2}}(eq^{-L}R)
$$

for sufficiently large n. In the same manner, it follows that

$$
\begin{aligned}
|I_6| &\le \sup_{|k|\le\lfloor n\delta\rfloor}|f_n(k+\lfloor nl\rfloor)-1|\sum_{k=-\lfloor n\delta\rfloor}^{\lfloor n\delta\rfloor} q^{k^2c_0+kb_n}|z|^k \\
&\le 2\eta_n(\delta)\Theta_{q^{c_0}}(q^{-L}R).
\end{aligned}
$$

The desired result (3.2) is obtained by a combination of the estimates for $I_1, \cdots, I_6$.

□

Proof of Theorem 3.1. Here we need to show that $r_n \to 0$ as $n \to \infty$. Let $0 < \varepsilon < c_0/2$, and choose $\delta = \delta(\varepsilon)$ as in condition (iv). We estimate I_1, I_2, I_3 and I_4 as before, and they all tend to zero as $n \to \infty$. As for I_5 and I_6, we also proceed as in Theorem 3.3, and obtain

$$\begin{aligned} |I_5| &\leq \varepsilon M |\ln q| \sum_{k=-\lfloor n\delta \rfloor}^{\lfloor n\delta \rfloor} k^2 e^{-\varepsilon k^2} q^{k^2 c_0 + k b_n} |z|^k \\ &\leq 4\varepsilon M |\ln q| \Theta_{q^{c_0/2}}(e q^{-L} R) \end{aligned}$$

and

$$|I_6| \leq 2\varepsilon \Theta_{q^{c_0}}(q^{-L} R).$$

Thus, $\lim_{n\to\infty} |r_n| \leq C\varepsilon$, where C is independent of ε. Since ε is arbitrary, the desired result (3.2) follows. □

3.2 The q-Airy function and the q-Airy polynomial

Recall from (1.4) that for $q \in (0,1)$,

$$(a;q)_0 := 1, \qquad (a;q)_n := \prod_{k=1}^{n} (1 - a q^{k-1}), \qquad n = 1, 2, \cdots, \infty.$$

We shall also make use of the identity

$$(q;q)_n = \frac{(q;q)_\infty}{(q^{n+1};q)_\infty}. \tag{3.4}$$

In this section, we investigate asymptotic behavior of the q-Airy function (1.8)

$$A_q(z) := \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k} (-z)^k$$

as $z \to \infty$, and the q-Airy polynomial (1.10)

$$A_{q,n}(z) := \sum_{k=0}^{n} \frac{q^{k^2}}{(q;q)_k} (-z)^k$$

as $n \to \infty$. As we shall see, our formulas involve the q-Theta function (1.9)

$$\Theta_q(z) := \sum_{k=-\infty}^{\infty} q^{k^2} z^k.$$

For convenience, we introduce the *half q-Theta function*:

$$\Theta_q^+(z) := \sum_{k=0}^{\infty} q^{k^2} z^k. \tag{3.5}$$

Clearly,

$$\Theta_q(z) + 1 = \Theta_q^+(z) + \Theta_q^+(1/z). \tag{3.6}$$

Proposition 3.4. *Let $z := q^{-nt}u$ with $u \neq 0$ and t being a fixed real number. When $t \geq 2$, we have*

$$A_{q,n}(z) = \frac{(-z)^n q^{n^2}}{(q;q)_\infty}\left[\Theta_q^+(-q^{-2n}/z) + O(q^{n(1-\delta)})\right] \tag{3.7}$$

uniformly for $|u| \geq 1/R$, where $\delta > 0$ is any small number and $R > 0$ is any large real number. When $0 < t < 2$, we have

$$A_{q,n}(z) = \frac{(-z)^m q^{m^2}}{(q;q)_\infty}\left[\Theta_q(-q^{2m}z) + O(q^{m(1-\delta)})\right], \tag{3.8}$$

where $m := \lfloor nt/2 \rfloor$ and $\delta > 0$ is any small number; this formula holds uniformly for $\frac{1}{R} \leq |u| \leq R$, where $R > 0$ is any large real number. When $t \leq 0$, we have

$$A_{q,n}(z) = A_q(z) + O(q^{n^2(1-\delta)}) \tag{3.9}$$

uniformly for $|u| \leq R$, where $\delta > 0$ is any small number. Furthermore, as $z \to \infty$, we have

$$A_q(z) = \frac{(-z)^m q^{m^2}}{(q;q)_\infty}\left[\Theta_q(-q^{2m}z) + O(q^{m(1-\delta)})\right], \tag{3.10}$$

where $m := \lfloor \frac{\ln|z|}{-2\ln q} \rfloor$ and $\delta > 0$ is any small number.

Proof. From the definition of q-Airy polynomial (1.10) we have

$$\begin{aligned} A_{q,n}(z) &= \sum_{k=0}^{n} \frac{q^{(n-k)^2}}{(q;q)_{n-k}}(-z)^{n-k} \\ &= \frac{(-z)^n q^{n^2}}{(q;q)_\infty} \sum_{k=0}^{n} q^{k^2}(q^{n-k+1};q)_\infty(-q^{-2n}/z)^k. \end{aligned}$$

If $t \geq 2$, we write

$$A_{q,n}(z) = \frac{(-z)^n q^{n^2}}{(q;q)_\infty} \left[\Theta_q^+(-q^{-2n}/z) + r_n(z) \right].$$

Then we have

$$\begin{aligned} r_n(z) &= \sum_{k=0}^{n} q^{k^2} (q^{n-k+1};q)_\infty (-q^{-2n}/z)^k - \sum_{k=0}^{\infty} q^{k^2} (-q^{-2n}/z)^k \\ &= I_1 + I_2 + I_3, \end{aligned}$$

where

$$\begin{aligned} I_1 &:= -\sum_{k=0}^{\lfloor n\delta \rfloor} q^{k^2} (1 - (q^{n-k+1};q)_\infty)(-q^{-2n}/z)^k, \\ I_2 &:= \sum_{k=\lfloor n\delta \rfloor + 1}^{n} q^{k^2} (q^{n-k+1};q)_\infty (-q^{-2n}/z)^k, \\ I_3 &:= -\sum_{k=\lfloor n\delta \rfloor + 1}^{\infty} q^{k^2} (-q^{-2n}/z)^k. \end{aligned}$$

On account of

$$1 - ab < (1-a) + (1-b)$$

for any $a, b \in (0,1)$ and by induction, it is verifiable that

$$1 - (q^{n-k+1};q)_\infty \leq \sum_{j=n-k+1}^{\infty} q^j = \frac{q^{n-k+1}}{1-q} \leq \frac{q^{n(1-\delta)}}{1-q}$$

for any $0 \leq k \leq \lfloor n\delta \rfloor$. Since

$$|q^{-2n}/z| = q^{n(t-2)}/|u| \leq R$$

for $t \geq 2$, we have

$$|I_1| \leq \frac{q^{n(1-\delta)}}{1-q} \sum_{k=0}^{\infty} q^{k^2} R^k = O(q^{n(1-\delta)}).$$

Furthermore, it is readily seen that

$$\begin{aligned}\max\{|I_2|,|I_3|\} &\leq \sum_{k=\lfloor n\delta\rfloor+1}^{\infty} q^{k^2}R^k \\ &= \sum_{k=0}^{\infty} q^{(k+\lfloor n\delta\rfloor+1)^2}R^{k+\lfloor n\delta\rfloor+1} \\ &\leq q^{n^2\delta^2}R^{n\delta+1}\Theta_q^+(q^{2n\delta}R) \\ &= O(q^{n^2\delta^2(1-\delta)}).\end{aligned}$$

From the above estimates we obtain

$$A_{q,n}(z) = \frac{(-z)^n q^{n^2}}{(q;q)_\infty}\left[\Theta_q^+(-q^{-2n}/z) + O(q^{n(1-\delta)})\right]$$

for any small $\delta > 0$. This proves (3.7).

Now, we consider the case $0 < t < 2$. Set $m := \lfloor nt/2\rfloor$; then we can rewrite the q-Airy polynomial (1.10) as

$$\begin{aligned}A_{q,n}(z) &= \sum_{k=0}^{n} \frac{q^{(k-m)^2-m^2}}{(q;q)_k}(-q^{2m}z)^k \\ &= \frac{(-z)^m q^{m^2}}{(q;q)_\infty}\sum_{k=-m}^{n-m} q^{k^2}(q^{k+m+1};q)_\infty(-q^{2m}z)^k. \qquad (3.11)\end{aligned}$$

To estimate the difference between the last sum and the q-Theta function, we let

$$\begin{aligned}r_n(z) &:= \frac{(q;q)_\infty}{(-z)^m q^{m^2}}A_{q,n}(z) - \Theta_q(-q^{2m}z) \\ &= \sum_{k=-m}^{n-m} q^{k^2}(q^{k+m+1};q)_\infty(-q^{2m}z)^k - \sum_{k=-\infty}^{\infty} q^{k^2}(-q^{2m}z)^k \\ &= I_1 + I_2 + I_3, \qquad (3.12)\end{aligned}$$

where

$$\begin{aligned}I_1 &:= \sum_{k=-\lfloor n\delta\rfloor}^{\lfloor n\delta\rfloor} q^{k^2}((q^{k+m+1};q)_\infty - 1)(-q^{2m}z)^k, \\ I_2 &:= \sum_{k=\lfloor n\delta\rfloor+1}^{n-m} q^{k^2}(q^{k+m+1};q)_\infty(-q^{2m}z)^k - \sum_{k=\lfloor n\delta\rfloor+1}^{\infty} q^{k^2}(-q^{2m}z)^k, \\ I_3 &:= \sum_{k=-m}^{-\lfloor n\delta\rfloor-1} q^{k^2}(q^{k+m+1};q)_\infty(-q^{2m}z)^k - \sum_{k=-\infty}^{-\lfloor n\delta\rfloor-1} q^{k^2}(-q^{2m}z)^k.\end{aligned}$$

Firstly, since $1/R \leq |u| \leq R$ and $-2 \leq 2m - nt \leq 0$, we have

$$q^2/R \leq |q^{2m}z| = q^{2m-nt}|u| \leq R/q^2.$$

On account of

$$1 - ab < (1-a) + (1-b)$$

for any $a, b \in (0,1)$ and by induction, we obtain

$$1 - (q^{k+m+1}; q)_\infty \leq \sum_{j=k+m+1}^{\infty} q^j = \frac{q^{k+m+1}}{1-q} \leq \frac{q^{m-n\delta}}{1-q}$$

for $-\lfloor n\delta \rfloor \leq k \leq \lfloor n\delta \rfloor$. Thus, it follows that

$$|I_1| \leq \frac{2q^{m-n\delta}}{1-q} \sum_{k=0}^{\infty} q^{k^2}(R/q^2)^k = O(q^{m-n\delta}). \tag{3.13}$$

Secondly, it can be shown that

$$\begin{aligned}
\max\{|I_2|, |I_3|\} &\leq 2 \sum_{k=\lfloor n\delta \rfloor+1}^{\infty} q^{k^2}(R/q^2)^k \\
&= 2 \sum_{k=0}^{\infty} q^{(k+\lfloor n\delta \rfloor+1)^2}(R/q^2)^{k+\lfloor n\delta \rfloor+1} \\
&= 2q^{(\lfloor n\delta \rfloor+1)^2}(R/q^2)^{\lfloor n\delta \rfloor+1}\Theta_q^+(q^{2\lfloor n\delta \rfloor}R) \\
&= O(q^{n^2\delta^2(1-\delta)}).
\end{aligned} \tag{3.14}$$

Finally, applying (3.13) and (3.14) to (3.12) gives

$$r_n(z) = O(q^{m-n\delta}).$$

Therefore,

$$A_{q,n}(z) = \frac{(-z)^m q^{m^2}}{(q;q)_\infty}\left[\Theta_q(-q^{2m}z) + O(q^{m-n\delta})\right]$$

for any small $\delta > 0$. Replacing δ by $\frac{t\delta}{2}$, formula (3.8) then follows since $m := \lfloor nt/2 \rfloor$.

When $t \leq 0$, we have $q^{-nt} \leq 1$ and hence $|z| = |q^{-nt}u| \leq R$. From (1.8) and (1.10) we obtain

$$
\begin{aligned}
|A_{q,n}(z) - A_q(z)| &= \left| \sum_{k=n+1}^{\infty} \frac{q^{k^2}}{(q;q)_k} (-z)^k \right| \\
&\leq \sum_{k=n+1}^{\infty} \frac{q^{k^2}}{(q;q)_\infty} R^k \\
&\leq \sum_{k=n}^{\infty} \frac{q^{k^2}}{(q;q)_\infty} R^k.
\end{aligned}
$$

For convenience, we have added a positive term in the last sum. Since the last sum can be expressed in terms of the half q-Theta function defined in (3.5), we have

$$
\begin{aligned}
|A_{q,n}(z) - A_q(z)| &\leq \sum_{l=0}^{\infty} \frac{q^{(l+n)^2}}{(q;q)_\infty} R^{l+n} \\
&= \frac{q^{n^2} R^n}{(q;q)_\infty} \Theta_q^+(q^{2n}R) \\
&= O(q^{n^2(1-\delta)})
\end{aligned}
$$

for any small $\delta > 0$. This ends the proof of (3.9).

The proof of (3.10) is similar to that of (3.8). Recall that $m := \lfloor \frac{\ln|z|}{-2\ln q} \rfloor$. When z tends to infinity, so does m. Furthermore, we have $1 \leq |q^{2m}z| \leq q^{-2}$. This suggests to change the variable in the q-Airy function from z into $q^{2m}z$. On account of (3.4),

$$
\begin{aligned}
A_q(z) &= \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k} (-z)^k \\
&= \sum_{k=-m}^{\infty} \frac{q^{(k+m)^2}}{(q;q)_{k+m}} (-z)^{k+m} \\
&= \frac{(-z)^m q^{m^2}}{(q;q)_\infty} \sum_{k=-m}^{\infty} q^{k^2} (q^{k+m+1};q)_\infty (-q^{2m}z)^k.
\end{aligned}
$$

To prove (3.10) we only need to estimate the remainder

$$
\begin{aligned}
r(z) &:= \frac{(q;q)_\infty}{(-z)^m q^{m^2}} A_q(z) - \Theta_q(-q^{2m}z) \\
&= \sum_{k=-m}^{\infty} q^{k^2}(q^{k+m+1};q)_\infty(-q^{2m}z)^k - \sum_{k=-\infty}^{\infty} q^{k^2}(-q^{2m}z)^k \\
&= I_1 + I_2 + I_3,
\end{aligned} \tag{3.15}
$$

where

$$
\begin{aligned}
I_1 &:= \sum_{k=-\lfloor m\delta\rfloor}^{\lfloor m\delta\rfloor} q^{k^2}((q^{k+m+1};q)_\infty - 1)(-q^{2m}z)^k, \\
I_2 &:= \sum_{k=\lfloor m\delta\rfloor+1}^{\infty} q^{k^2}(q^{k+m+1};q)_\infty(-q^{2m}z)^k - \sum_{k=\lfloor m\delta\rfloor+1}^{\infty} q^{k^2}(-q^{2m}z)^k, \\
I_3 &:= \sum_{k=-m}^{-\lfloor m\delta\rfloor-1} q^{k^2}(q^{k+m+1};q)_\infty(-q^{2m}z)^k - \sum_{k=-\infty}^{-\lfloor m\delta\rfloor-1} q^{k^2}(-q^{2m}z)^k.
\end{aligned}
$$

Again since $q^2 < 1 \le |q^{2m}z| \le q^{-2}$, similar to the proof of (3.8) one can show that for any fixed small $\delta > 0$,

$$
|I_1| \le 2\frac{q^{m-m\delta}}{1-q}\sum_{k=0}^{\infty} q^{k^2}q^{-2k} = O(q^{m(1-\delta)}), \tag{3.16}
$$

and

$$
\begin{aligned}
\max\{|I_2|,|I_3|\} &\le 2\sum_{k=\lfloor m\delta\rfloor+1}^{\infty} q^{k^2}q^{-2k} \\
&= 2q^{\lfloor m\delta\rfloor^2-1}\Theta_q^+(q^{2\lfloor m\delta\rfloor}) \\
&= O(q^{m^2\delta^2(1-\delta)}).
\end{aligned} \tag{3.17}
$$

A combination of (3.15), (3.16) and (3.17) gives (3.10). This ends our proof. □

3.3 Uniform asymptotic formulas for some q-orthogonal polynomials

In this section, we derive several uniform asymptotic formulas for the q^{-1}-Hermite polynomials (1.5), the Stieltjes-Wigert polynomials (1.6), and the q-

Laguerre polynomials (1.7). We use the same scale as in [15]. Thus, for the q^{-1}-Hermite polynomials, we set

$$z = \sinh\xi := (q^{-nt}u - q^{nt}u^{-1})/2$$

with $u \neq 0$ and $t \geq 0$. For the Stieltjes-Wigert polynomials and the q-Laguerre polynomials, we set

$$z := q^{-nt}u$$

with $u \neq 0$ and $t \geq 1$. After rescaling, we have

$$h_n(\sinh\xi|q) = u^n q^{-n^2 t}\sum_{k=0}^{n}\frac{(q^{n-k+1};q)_k}{(q;q)_k}q^{k^2}(-u^{-2}q^{n(2t-1)})^k, \tag{3.18}$$

$$S_n(z;q) = \frac{(-u)^n q^{n^2(1-t)}}{(q;q)_n}\sum_{k=0}^{n}\frac{(q^{n-k+1};q)_k}{(q;q)_k}q^{k^2}(-u^{-1}q^{n(t-2)})^k, \tag{3.19}$$

$$L_n^\alpha(z;q) = \frac{(-uq^\alpha)^n q^{n^2(1-t)}}{(q;q)_n} \times\sum_{k=0}^{n}\frac{(q^{\alpha+1+n-k};q)_k(q^{n-k+1};q)_k}{(q;q)_k}q^{k^2}(-u^{-1}q^{n(t-2)-\alpha})^k. \tag{3.20}$$

Theorem 3.5. *Let $z = \sinh\xi := (q^{-nt}u - q^{nt}u^{-1})/2$ with $u \in \mathbb{C}$ and $|u| \geq 1/R$, where $R > 0$ is any fixed large number. Given any small $\delta > 0$, we have*

$$h_n(\sinh\xi|q) = u^n q^{-n^2 t}\left[A_{q,n}(u^{-2}q^{n(2t-1)}) + r_n(t,u)\right] \tag{3.21}$$

for $t > 1/2 - \delta$, where the remainder satisfies

$$|r_n(t,u)| \leq \frac{q^{n(1-3\delta)}}{1-q}A_{q,n}(-|u|^{-2}q^{n(2t-1)}) + \frac{q^{3n^2\delta^2-2n\delta}R^{2(\lfloor 3n\delta\rfloor+1)}}{(q;q)_\infty}\Theta_q^+(q^{4n\delta}R^2). \tag{3.22}$$

On the other hand, for $0 \leq t < 1/2$, we have

$$h_n(\sinh\xi|q) = \frac{(-1)^m u^{n-2m}q^{-n^2t-m[n(1-2t)-m]}}{(q;q)_\infty} \times\left[\Theta_q(-u^{-2}q^{2m-n(1-2t)}) + O(q^{n(l-\delta)})\right], \tag{3.23}$$

where $l := 1/2 - t$, $m := \lfloor nl \rfloor$ and $\delta > 0$ is any small number. The O-term in (3.23) is uniform with respect to $u \in T_R := \{z \in \mathbb{C} : R^{-1} \le |z| \le R\}$ with $R > 0$ being any large real number.

Proof. From (1.10), (3.18) and (3.21) it is easily seen that

$$r_n(t,u) = \sum_{k=0}^{n} \frac{(q^{n-k+1};q)_k - 1}{(q;q)_k} q^{k^2}(-u^{-2}q^{n(2t-1)})^k = I_1 + I_2, \tag{3.24}$$

where

$$\begin{aligned} I_1 &:= \sum_{k=0}^{\lfloor n\delta_1 \rfloor} \frac{(q^{n-k+1};q)_k - 1}{(q;q)_k} q^{k^2}(-u^{-2}q^{n(2t-1)})^k, \\ I_2 &:= \sum_{k=\lfloor n\delta_1 \rfloor+1}^{n} \frac{(q^{n-k+1};q)_k - 1}{(q;q)_k} q^{k^2}(-u^{-2}q^{n(2t-1)})^k. \end{aligned}$$

Here $\delta_1 \in (0,1)$ is a small number to be determined later. For any $0 \le k \le \lfloor n\delta_1 \rfloor$,

$$0 \le 1 - (q^{n-k+1};q)_k < \frac{q^{n-k+1}}{1-q} \le \frac{q^{n(1-\delta_1)}}{1-q}.$$

Thus,

$$\begin{aligned} |I_1| &\le \sum_{k=0}^{n} \frac{q^{n(1-\delta_1)}}{1-q} \frac{q^{k^2}}{(q;q)_k} (|u|^{-2}q^{n(2t-1)})^k \\ &= \frac{q^{n(1-\delta_1)}}{1-q} A_{q,n}(-|u|^{-2}q^{n(2t-1)}). \end{aligned} \tag{3.25}$$

Furthermore, since

$$0 \le 1 - (q^{n-k+1};q)_k \le 1$$

for any $k \in [0,n]$ and

$$|u|^{-2}q^{n(2t-1)} \le q^{-2n\delta}R^2$$

for $t > 1/2 - \delta$, we obtain

$$\begin{aligned} |I_2| &\le \sum_{k=\lfloor n\delta_1 \rfloor+1}^{\infty} \frac{q^{k^2}}{(q;q)_\infty} (q^{-2n\delta}R^2)^k \\ &= \frac{q^{(\lfloor n\delta_1 \rfloor+1)^2 - 2n\delta(\lfloor n\delta_1 \rfloor+1)} R^{2(\lfloor n\delta_1 \rfloor+1)}}{(q;q)_\infty} \Theta_q^+(q^{2(\lfloor n\delta_1 \rfloor+1)-2n\delta}R^2) \\ &\le \frac{q^{n^2\delta_1^2 - 2n\delta(n\delta_1+1)} R^{2(\lfloor n\delta_1 \rfloor+1)}}{(q;q)_\infty} \Theta_q^+(q^{2n\delta_1-2n\delta}R^2). \end{aligned} \tag{3.26}$$

Choose $\delta_1 := 3\delta$. Then (3.22) follows from (3.24), (3.25) and (3.26).

When $0 \le t < 1/2$, we apply Theorem 3.3 to (3.18) with

$$l = 1/2 - t, \quad m = \lfloor nl \rfloor, \quad g_n(k) = k^2 - 2nlk + m(2nl - m),$$
$$f_n(k) = (q^{k+1}; q)_\infty (q^{n-k+1}; q)_k, \quad z = -u^2, \quad c_0 = 1, \quad M = 1,$$
$$b_n = 2(m - nl), \quad L = 2, \quad A_\delta = \delta^2(1-\delta), \quad \eta_n(\delta) = 2q^{n(l-\delta)}/(1-q).$$

To verify condition (ii) in Theorem 3.1 (which is also assumed in Theorem 3.3), we choose $N(\delta) = \lfloor 2/\delta^2 \rfloor$. Then it is readily seen that for $k \in [0, n(l-\delta)] \cup [n(l+\delta)-1, n]$ and $n > N(\delta)$, we have

$$\begin{aligned} g_n(k) &= (k - nl)^2 - (m - nl)^2 \\ &> (n\delta - 1)^2 - 1 \\ &> n^2\delta^2(1-\delta) \\ &= n^2 A_\delta. \end{aligned} \tag{3.27}$$

To show that condition (iv′) in Theorem 3.3 also holds, we first note that

$$1 - ab < (1-a) + (1-b)$$

for $0 < a, b < 1$, and hence

$$\begin{aligned} |f_n(k) - 1| &= 1 - (q^{k+1}; q)_\infty (q^{n-k+1}; q)_\infty \\ &< 1 - (q^{k+1}; q)_\infty + 1 - (q^{n-k+1}; q)_k. \end{aligned}$$

For any positive integers m and k, we have by induction

$$\begin{aligned} 1 - (q^m; q)_k &< q^m + q^{m+1} + \cdots + q^{m+k-1} \\ &= \frac{q^m - q^{m+k}}{1-q} \\ &< \frac{q^m}{1-q}. \end{aligned} \tag{3.28}$$

Letting $k \to \infty$ yields

$$1 - (q^m; q)_\infty \le \frac{q^m}{1-q}.$$

Thus,

$$|f_n(k) - 1| \leq \frac{q^{k+1}}{1-q} + \frac{q^{n-k+1}}{1-q} \leq \frac{2q^{n(l-\delta)}}{1-q} = \eta_n(\delta) \tag{3.29}$$

for $k \in [n(l-\delta), n(l+\delta)]$, where we have used $l = 1/2 - t \leq 1/2$. Next, we observe that

$$\begin{aligned} g_n(k) &= k^2 - 2nlk + m(2nl - m) \\ &= 2(k-m)(m-nl) + (k-m)^2. \end{aligned}$$

With $c_0 = 1$, $m = \lfloor nl \rfloor$ and $b_n = 2(m - nl)$, the last equation becomes

$$g_n(k) = (k - \lfloor nl \rfloor)b_n + c_0(k - \lfloor nl \rfloor)^2, \tag{3.30}$$

thus establishing condition (iv′). Formula (3.23) now follows from (3.2) and (3.3). □

Theorem 3.6. *Let $z := q^{-nt}u$ with $t \geq 1$, $u \in \mathbb{C}$ and $|u| \geq 1/R$, where $R > 0$ is any fixed large number. Given any small $\delta > 0$, we have*

$$S_n(z;q) = \frac{(-u)^n q^{n^2(1-t)}}{(q;q)_n}\left[A_{q,n}(u^{-1}q^{n(t-2)}) + r_n(t,u)\right] \tag{3.31}$$

for $t > 2(1-\delta)$, where the remainder satisfies

$$\begin{aligned} |r_n(t,u)| \leq & \frac{q^{n(1-3\delta)}}{1-q} A_{q,n}(-|u|^{-1}q^{n(t-2)}) \\ & + \frac{q^{3n^2\delta^2 - 2n\delta} R^{\lfloor 3n\delta \rfloor + 1}}{(q;q)_\infty} \Theta_q^+(q^{4n\delta}R). \end{aligned} \tag{3.32}$$

When $1 \leq t < 2$, we have

$$\begin{aligned} S_n(z;q) = & \frac{(-u)^{n-m} q^{n^2(1-t) - m[n(2-t)-m]}}{(q;q)_n (q;q)_\infty} \\ & \times \left[\Theta_q(-u^{-1}q^{2m-n(2-t)}) + O(q^{n(l-\delta)})\right], \end{aligned} \tag{3.33}$$

where $l := 1 - t/2$, $m := \lfloor nl \rfloor$ and $\delta > 0$ is any small number. This asymptotic formula holds uniformly for $u \in T_R := \{z \in \mathbb{C} : R^{-1} \leq |z| \leq R\}$, where $R > 0$ is any large real number.

Proof. On account of (1.10) and (3.19), we obtain from (3.31) that

$$\begin{aligned} r_n(z) :=& \frac{(q;q)_n}{(-u)^n q^{n^2(1-t)}} S_n(z;q) - A_{q,n}(u^{-1}q^{n(t-2)}) \\ =& -\sum_{k=0}^{n} \frac{1-(q^{n-k+1};q)_k}{(q;q)_k} q^{k^2}(-u^{-1}q^{n(t-2)})^k \\ =& -I_1 - I_2, \end{aligned} \tag{3.34}$$

where

$$\begin{aligned} I_1 :=& \sum_{k=0}^{\lfloor n\delta_1 \rfloor} \frac{1-(q^{n-k+1};q)_k}{(q;q)_k} q^{k^2}(-u^{-1}q^{n(t-2)})^k, \\ I_2 :=& \sum_{k=\lfloor n\delta_1 \rfloor+1}^{n} \frac{1-(q^{n-k+1};q)_k}{(q;q)_k} q^{k^2}(-u^{-1}q^{n(t-2)})^k. \end{aligned}$$

Here $\delta_1 \in (0,1)$ is a small number to be determined later. In view of the inequality

$$1 - ab < (1-a) + (1-b)$$

for any $a, b \in (0,1)$ and by induction, we can show that for any $0 \le k \le n\delta_1$,

$$1-(q^{n-k+1};q)_k < \sum_{i=1}^{k} q^{n-k+i} < \sum_{i=0}^{\infty} q^{n-k+i} = \frac{q^{n-k}}{1-q} \le \frac{q^{n(1-\delta_1)}}{1-q}.$$

Thus, from the definition of q-Airy polynomial (1.10) we obtain

$$\begin{aligned} |I_1| \le& \sum_{k=0}^{\lfloor n\delta_1 \rfloor} \frac{q^{n(1-\delta_1)}}{1-q} \frac{q^{k^2}}{(q;q)_k} \left|u^{-1}q^{n(t-2)}\right|^k \\ \le& \frac{q^{n(1-\delta_1)}}{1-q} A_{q,n}(-|u|^{-1}q^{n(t-2)}). \end{aligned} \tag{3.35}$$

Furthermore, since $0 < 1-(q^{n-k+1};q)_k < 1$ for any $n\delta_1 \le k \le n$, it follows that

$$\begin{aligned} |I_2| \le& \sum_{k=\lfloor n\delta_1 \rfloor+1}^{n} \frac{q^{k^2}}{(q;q)_\infty} \left|u^{-1}q^{n(t-2)}\right|^k \\ =& \sum_{k=0}^{n-\lfloor n\delta_1 \rfloor-1} \frac{q^{(k+\lfloor n\delta_1 \rfloor+1)^2}}{(q;q)_\infty} \left|u^{-1}q^{n(t-2)}\right|^{k+\lfloor n\delta_1 \rfloor+1} \\ \le& \sum_{k=0}^{\infty} \frac{q^{(k+n\delta_1)^2}}{(q;q)_\infty} |u^{-1}q^{n(t-2)}|^{k+\lfloor n\delta_1 \rfloor+1} \\ =& \frac{q^{n^2\delta_1^2}\left|u^{-1}q^{n(t-2)}\right|^{\lfloor n\delta_1 \rfloor+1}}{(q;q)_\infty} \Theta_q^+(q^{2n\delta_1}|u^{-1}q^{n(t-2)}|). \end{aligned}$$

By virtue of $t > 2(1-\delta)$ and $|u| \geq 1/R$, we have $|u^{-1}q^{n(t-2)}| \leq q^{-2n\delta}R$, and thus

$$\begin{aligned}|I_2| &\leq \frac{q^{n^2\delta_1^2}q^{-2n\delta(\lfloor n\delta_1\rfloor+1)}R^{\lfloor n\delta_1\rfloor+1}}{(q;q)_\infty}\Theta_q^+(q^{2n(\delta_1-\delta)}R)\\ &\leq \frac{q^{n^2\delta_1^2-2n\delta(n\delta_1+1)}R^{\lfloor n\delta_1\rfloor+1}}{(q;q)_\infty}\Theta_q^+(q^{2n(\delta_1-\delta)}R). \qquad (3.36)\end{aligned}$$

Set $\delta_1 := 3\delta$. A combination of (3.34), (3.35) and (3.36) gives (3.32) immediately.

For $1 \leq t < 2$, we apply Theorem 3.3 to (3.31) with

$$\begin{gathered} l = 1 - t/2, \quad m = \lfloor nl \rfloor, \quad g_n(k) = k^2 - 2nlk + m(2nl - m),\\ f_n(k) = (q^{n-k+1};q)_k(q^{k+1};q)_\infty, \quad z = -u^{-1}, \quad c_0 = 1, \quad M = 1,\\ b_n = 2(m - nl), \quad L = 2, \quad A_\delta = \delta^2(1-\delta), \quad \eta_n(\delta) = 2q^{n(l-\delta)}/(1-q). \end{gathered}$$

The arguments for verifying the conditions in Theorems 3.1 and 3.3 are the same as those used in the proof of Theorem 3.5. □

Theorem 3.7. *Assume that α is real and $\alpha > -1$. Let $z := q^{-nt}u$ with $u \in \mathbb{C}$ and $|u| \geq 1/R$, where $R > 0$ is any fixed large number. Given any small $\delta > 0$, we have*

$$L_n^\alpha(z;q) = \frac{(-uq^\alpha)^n q^{n^2(1-t)}}{(q;q)_n}\left[A_{q,n}(u^{-1}q^{n(t-2)-\alpha}) + r_n(t,u)\right] \qquad (3.37)$$

for $t > 2(1-\delta)$, where the remainder satisfies

$$\begin{aligned}|r_n(t,u)| \leq& \frac{2q^{n(1-2\delta)}}{1-q}A_{q,n}(-|u|^{-1}q^{n(t-2)-\alpha})\\ &+\frac{q^{3n^2\delta^2-2n\delta}(q^{-\alpha}R)^{\lfloor 3n\delta\rfloor+1}}{(q;q)_\infty}\Theta_q^+(q^{4n\delta-\alpha}R). \qquad (3.38)\end{aligned}$$

On the other hand, when $1 \leq t < 2$, we have

$$\begin{aligned}L_n^\alpha(z;q) =& \frac{(-uq^\alpha)^{n-m}q^{n^2(1-t)-m[n(2-t)-m]}}{(q;q)_n(q;q)_\infty}\\ &\times\left[\Theta_q(-u^{-1}q^{2m-n(2-t)-\alpha}) + O(q^{n(l-\delta)})\right], \qquad (3.39)\end{aligned}$$

where $l := 1 - t/2$, $m := \lfloor nl \rfloor$ and $\delta > 0$ is any small number. The asymptotic formula holds uniformly for $u \in T_R := \{z \in \mathbb{C} : R^{-1} \leq |z| \leq R\}$, where $R > 0$ is any large real number.

Proof. It follows from (1.10) and (3.20) that the remainder in (3.37) can be written as

$$\begin{aligned} r_n(z) &= \sum_{k=0}^{n} \frac{(q^{\alpha+1+n-k};q)_k(q^{n-k+1};q)_k - 1}{(q;q)_k} q^{k^2}(-u^{-1}q^{n(t-2)-\alpha})^k \\ &= I_1 + I_2, \end{aligned} \tag{3.40}$$

where

$$\begin{aligned} I_1 &:= \sum_{k=0}^{\lfloor n\delta_1 \rfloor} \frac{(q^{\alpha+1+n-k};q)_k(q^{n-k+1};q)_k - 1}{(q;q)_k} q^{k^2}(-u^{-1}q^{n(t-2)-\alpha})^k, \\ I_2 &:= \sum_{k=\lfloor n\delta_1 \rfloor+1}^{n} \frac{(q^{\alpha+1+n-k};q)_k(q^{n-k+1};q)_k - 1}{(q;q)_k} q^{k^2}(-u^{-1}q^{n(t-2)-\alpha})^k. \end{aligned}$$

Here $\delta_1 \in (0,1)$ is a small number to be determined later. Since

$$1 - ab < (1-a) + (1-b)$$

for any $a, b \in (0,1)$, we have for any $0 \le k \le \lfloor n\delta_1 \rfloor$,

$$1 - (q^{\alpha+1+n-k};q)_k(q^{n-k+1};q)_k < 1 - (q^{\alpha+1+n-k};q)_k + 1 - (q^{n-k+1};q)_k.$$

In view of (3.28), it follows that

$$1 - (q^{\alpha+1+n-k};q)_k(q^{n-k+1};q)_k < \frac{q^{\alpha+1+n-k} + q^{n-k+1}}{1-q} < \frac{2q^{n(1-\delta_1)}}{1-q},$$

where we have used the assumption $\alpha > -1$. Therefore,

$$\begin{aligned} |I_1| &\le \sum_{k=0}^{n} \frac{2q^{n(1-\delta_1)}}{1-q} \frac{q^{k^2}}{(q;q)_k} (|u|^{-1}q^{n(t-2)-\alpha})^k \\ &= \frac{2q^{n(1-\delta_1)}}{1-q} A_{q,n}(-|u|^{-1}q^{n(t-2)-\alpha}). \end{aligned} \tag{3.41}$$

Furthermore, since

$$0 \le 1 - (q^{\alpha+1+n-k};q)_k(q^{n-k+1};q)_k \le 1$$

for any $k \in [0,n]$, and

$$|u|^{-1}q^{n(t-2)-\alpha} \le q^{-2n\delta-\alpha}R$$

for $t > 2(1-\delta)$, we obtain

$$\begin{aligned}
|I_2| &\le \sum_{k=\lfloor n\delta_1\rfloor+1}^{\infty} \frac{q^{k^2}}{(q;q)_\infty}(q^{-2n\delta-\alpha}R)^k \\
&= \frac{q^{(\lfloor n\delta_1\rfloor+1)^2-2n\delta(\lfloor n\delta_1\rfloor+1)}(q^{-\alpha}R)^{\lfloor n\delta_1\rfloor+1}}{(q;q)_\infty}\Theta_q^+(q^{2(\lfloor n\delta_1\rfloor+1)-2n\delta-\alpha}R) \\
&\le \frac{q^{n^2\delta_1^2-2n\delta(n\delta_1+1)}(q^{-\alpha}R)^{\lfloor n\delta_1\rfloor+1}}{(q;q)_\infty}\Theta_q^+(q^{2n\delta_1-2n\delta-\alpha}R). \qquad (3.42)
\end{aligned}$$

Set $\delta_1 := 3\delta$, then (3.38) follows from (3.40), (3.41) and (3.42).

When $1 \le t < 2$, we apply Theorem 3.3 to (3.20) with

$$l = 1 - t/2, \quad m = \lfloor nl\rfloor, \quad g_n(k) = k^2 - 2nlk + m(2nl-m),$$

$$f_n(k) = (q^{\alpha+1+n-k};q)_k(q^{n-k+1};q)_k(q^{k+1};q)_\infty, \quad z = -u^{-1}q^{-\alpha}, \quad c_0 = 1, \quad M = 1,$$

$$b_n = 2(m-nl), \quad L = 2, \quad A_\delta = \delta^2(1-\delta), \quad \eta_n(\delta) = 3q^{n(l-\delta)}/(1-q).$$

The verification of condition (iv$'$) in Theorem 3.3 proceeds along the same lines as that given in Theorem 3.5. In particular, since

$$|f_n(k)-1| = 1 - (q^{\alpha+1+n-k};q)_k(q^{n-k+1};q)_k(q^{k+1};q)_\infty$$

and

$$1 - abc < (1-a) + (1-b) + (1-c)$$

for $0 < a, b, c < 1$, we have

$$|f_n(k)-1| \le 1 - (q^{\alpha+1+n-k};q)_k + 1 - (q^{n-k+1};q)_k + 1 - (q^{k+1};q)_\infty.$$

Thus, by (3.28)

$$|f_n(k)-1| \le \frac{q^{\alpha+1+n-k}+q^{n-k+1}+q^{k+1}}{1-q} \le \frac{3q^{n(l-\delta)}}{1-q} = \eta_n(\delta)$$

for $k \in [n(l-\delta), n(l+\delta)]$. $\square$

Chapter 4

Appendix

4.1 Explicit formulas of some integrals

In this section we calculate some integrals which are frequently used in this thesis.

Proposition 4.1. *Let a and b be two constants such that $0 < a < 1 < b$ and $ab = 1$. For any $x \in [a, b]$, we have*

$$I_1(x) := \int_{ax}^{1} \frac{ds}{\sqrt{(bs-x)(x-as)}} = \arccos \frac{(b+a)x-2}{(b-a)x}, \tag{4.1}$$

$$\begin{aligned} I_2(x) &:= \int_a^x \arccos \frac{(b+a)s-2}{(b-a)s} ds \\ &= x \arccos \frac{(b+a)x-2}{(b-a)x} - \arccos \frac{2x-(b+a)}{b-a} + \pi(1-a), \end{aligned} \tag{4.2}$$

$$\begin{aligned} I_3(x) &:= \int_{ax}^1 \arccos \frac{2x-(b+a)s}{(b-a)s} ds \\ &= \arccos \frac{2x-(b+a)}{b-a} - x \arccos \frac{(b+a)x-2}{(b-a)x}. \end{aligned} \tag{4.3}$$

Especially when $x = b$, we have

$$I_2(b) = \int_a^b \arccos \frac{(b+a)s-2}{(b-a)s} ds = \pi(1-a). \tag{4.4}$$

For any $z \in \mathbb{C} \setminus [a, b)$, we have

$$\begin{aligned} I_4(z) &:= \frac{1}{\pi} \int_a^b \frac{1}{z-s} \arccos \frac{(b+a)s-2}{(b-a)s} ds \\ &= -\log \frac{z(b+a)-2+2\sqrt{(z-a)(z-b)}}{(z-a)(1+a)(1+b)}. \end{aligned} \tag{4.5}$$

Especially when $z = b$, we have

$$I_4(b) = \frac{1}{\pi} \int_a^b \frac{1}{b-s} \arccos \frac{(b+a)s-2}{(b-a)s} ds = 2\log(1+a). \tag{4.6}$$

For any $z \in \mathbb{C} \setminus [0, b)$, we have

$$I_5(z) := \int_0^a \frac{\sqrt{(z-a)(z-b)}}{\sqrt{(a-s)(b-s)}} \frac{ds}{s-z}$$
$$= -\log \frac{z(b+a) - 2 + 2\sqrt{(z-a)(z-b)}}{z(b-a)}. \tag{4.7}$$

Finally, we have

$$I_6 := \int_a^b \log(b-s) \arccos \frac{(b+a)s - 2}{(b-a)s} ds$$
$$= \pi[2b \log(1+a) + (1-a)\log(b-a) - 2\log 2 - 1 + a]. \tag{4.8}$$

Proof. By a change of variable $s = xt$ we have

$$I_1(x) = \int_a^{1/x} \frac{ds}{\sqrt{(bs-1)(1-as)}}.$$

Set $y := 1/x$. On account of $ab = 1$, the equality (4.1) is the same as

$$\int_a^y \frac{ds}{\sqrt{(s-a)(b-s)}} = \arccos \frac{(b+a) - 2y}{b-a},$$

which is obvious since the functions on both side have the same derivative on y and the same value at $y = a$. This proves (4.1).

The equality (4.2) can be proved in the same manner. We observe that the functions on the both side of (4.2) have the same derivative on x. Moreover, they all vanish at the point $x = a$. This proves (4.2).

To prove (4.3), we make a change of variable $s = 1/t$. Then we obtain

$$I_3(x) = x \int_a^{1/x} \arccos \frac{2 - (b+a)t}{(b-a)t} dt = -x \int_a^{1/x} \left[\arccos \frac{(b+a)t - 2}{(b-a)t} - \pi \right] dt.$$

On account of (4.2) we have

$$I_3(x) = -x \left[\frac{1}{x} \arccos \frac{(b+a) - 2x}{b-a} - \arccos \frac{2 - (b+a)x}{(b-a)x} + \pi - \pi a - \pi(1/x - a) \right]$$
$$= \arccos \frac{2x - (b+a)}{b-a} - x \arccos \frac{(b+a)x - 2}{(b-a)x}.$$

This gives (4.3).

An integration by parts gives

$$
\begin{aligned}
I_4(z) :=& \frac{1}{\pi}\int_a^b \frac{1}{z-s}\arccos\frac{s(b+a)-2}{s(b-a)}ds \\
=& \frac{-\log(z-s)}{\pi}\arccos\frac{s(b+a)-2}{s(b-a)}\Big|_a^b - \int_a^b \frac{\log(z-s)ds}{\pi s\sqrt{(s-a)(b-s)}} \\
=& \log(z-a) + \int_{z-b}^{z-a} \frac{\log s ds}{\pi(s-z)\sqrt{[s-(z-b)]\cdot[(z-a)-s]}} \\
=& \log(z-a) + \log\frac{[z+1+\sqrt{(z-a)(z-b)}]^2}{[\sqrt{z-a}+\sqrt{z-b}]^2 z^2}, \qquad (4.9)
\end{aligned}
$$

where we have used the equality (cf. [23, Lemma 2, (2.46)])

$$
\begin{aligned}
&\int_\alpha^\beta \frac{\log s ds}{\pi(s-z)\sqrt{(s-\alpha)(\beta-s)}} \\
=& \frac{1}{\sqrt{(z-\alpha)(z-\beta)}}\log\frac{[z+\sqrt{\alpha\beta}+\sqrt{(z-\alpha)(z-\beta)}]^2}{(\sqrt{\alpha}+\sqrt{\beta})^2 z^2} \qquad (4.10)
\end{aligned}
$$

with $\alpha = z-b$ and $\beta = z-a$. To show that (4.9) is the same as (4.5), we obtain from the relation $ab = 1$ that

$$
\begin{aligned}
& z(b+a) - 2 + 2\sqrt{(z-a)(z-b)} \\
=& \left[z+1-\sqrt{(z-a)(z-b)}\right]\left[z+\sqrt{(z-a)(z-b)}-1\right] \\
=& \frac{(1+a)(1+b)z^2[\sqrt{z-a}+\sqrt{z-b}]^2}{[z+1+\sqrt{(z-a)(z-b)}]^2}. \qquad (4.11)
\end{aligned}
$$

Applying (4.11) to (4.9) gives (4.5) immediately.

A change of variable $x = \frac{b+a}{2} - \frac{b-a}{2}t$ gives

$$
\int_0^a \frac{1}{\sqrt{(a-x)(b-x)}}\frac{dx}{x-z} = \frac{-2}{b-a}\int_1^{\frac{b+a}{b-a}} \frac{1}{\sqrt{t^2-1}}\frac{dt}{t+w}, \qquad (4.12)
$$

where $w := \frac{2z-(b+a)}{b-a}$. Now, we make another change of variable $t = \frac{1}{2}(s+\frac{1}{s})$ with $s \geq 1$. The right-hand side of (4.12) becomes

$$
\frac{-2}{b-a}\int_1^\lambda \frac{2ds}{s^2+1+2ws} = \frac{-2}{b-a}\cdot\frac{1}{\sqrt{w^2-1}}\cdot\log\frac{(\lambda-s_+)(1-s_-)}{(\lambda-s_-)(1-s_+)}, \qquad (4.13)
$$

where $\lambda > 1$ is a solution to the equation

$$
\frac{b+a}{b-a} = \frac{1}{2}(\lambda + \frac{1}{\lambda}),
$$

and $s_\pm := -w \pm \sqrt{w^2-1}$ are the roots of the equation $s^2+1+2ws=0$. Recall $w := \frac{2z-(b+a)}{b-a}$, it is easily seen that

$$\frac{w+1}{w-1} = \frac{z-a}{z-b}, \qquad \frac{\lambda-1}{\lambda+1} = \sqrt{\frac{a}{b}}. \tag{4.14}$$

Moreover, we have

$$\sqrt{w^2-1} = \frac{2}{b-a}\sqrt{(z-a)(z-b)}.$$

Thus, it follows from (4.12) and (4.13) that

$$I_5(z) := \int_0^a \frac{\sqrt{(z-a)(z-b)}}{\sqrt{(a-x)(b-x)}}\frac{dx}{x-z} = -\log\frac{(\lambda-s_+)(1-s_-)}{(\lambda-s_-)(1-s_+)}. \tag{4.15}$$

Since $s_\pm := -w \pm \sqrt{w^2-1}$, we have

$$\frac{(\lambda-s_+)(1-s_-)}{(\lambda-s_-)(1-s_+)} = \frac{\lambda-\lambda s_- - s_+ + 1}{\lambda - \lambda s_+ - s_- + 1} = \frac{(\lambda+1)(w+1)+(\lambda-1)\sqrt{w^2-1}}{(\lambda+1)(w+1)-(\lambda-1)\sqrt{w^2-1}}.$$

On account of (4.14) the fact $ab=1$, the right-hand side of the last equation becomes

$$\frac{\sqrt{b}\sqrt{z-a}+\sqrt{a}\sqrt{z-b}}{\sqrt{b}\sqrt{z-a}-\sqrt{a}\sqrt{z-b}} = \frac{(b+a)z-2+2\sqrt{(z-a)(z-b)}}{(b-a)z}.$$

Coupling this with (4.15) gives (4.7).

We now prove (4.8). Firstly, from the equality (4.1), the relation $ab=1$ and Fubini's theorem we obtain

$$\begin{aligned} I_6 &= \int_a^b \int_{ax}^1 \frac{\log(b-x)dsdx}{\sqrt{(bs-x)(x-as)}} \\ &= \int_{a^2}^1 \int_a^{bs} \frac{\log(b-x)dxds}{\sqrt{(bs-x)(x-as)}}. \end{aligned}$$

An integration by parts gives

$$\begin{aligned} \int_a^{bs} \frac{\log(b-x)dx}{\sqrt{(bs-x)(x-as)}} &= \log(b-a)\arccos\frac{2a-(b+a)s}{(b-a)s} \\ &\quad + \int_a^{bs}\frac{1}{x-b}\arccos\frac{2x-(b+a)s}{(b-a)s}dx. \end{aligned}$$

Hence, we can write $I_6 = I_{61} + I_{62}$, where

$$I_{61} := \int_{a^2}^{1} \log(b-a) \arccos \frac{2a-(b+a)s}{(b-a)s} ds,$$
$$I_{62} := \int_{a^2}^{1} \int_{a}^{bs} \frac{1}{x-b} \arccos \frac{2x-(b+a)s}{(b-a)s} dxds.$$

By a change of variable $s = at$, we obtain from (4.4) and the relation $ab = 1$ that

$$\begin{aligned} I_{61} &= a \log(b-a) \int_a^b \arccos \frac{2-(b+a)t}{(b-a)t} dt \\ &= a \log(b-a) \int_a^b \left[\pi - \arccos \frac{(b+a)t-2}{(b-a)t} \right] dt \\ &= [a \log(b-a)] \cdot [\pi(b-a) - \pi(1-a)] \\ &= \pi(1-a) \log(b-a). \end{aligned} \tag{4.16}$$

Now, we are left to calculate I_{62}. Fubini's theorem together with (4.3) gives

$$\begin{aligned} I_{62} &= \int_a^b \int_{ax}^{1} \frac{1}{x-b} \arccos \frac{2x-(b+a)s}{(b-a)s} dsdx \\ &= \int_a^b \frac{1}{x-b} \arccos \frac{2x-(b+a)}{b-a} dx - \int_a^b \frac{x}{x-b} \arccos \frac{(b+a)x-2}{(b-a)x} dx. \end{aligned} \tag{4.17}$$

On the one hand, a change of variable $x = \frac{b-a}{2}t + \frac{b+a}{2}$ yields

$$\int_a^b \frac{1}{x-b} \arccos \frac{2x-(b+a)}{b-a} dx = \int_{-1}^{1} \frac{\arccos t}{t-1} dt = -2\pi \log 2, \tag{4.18}$$

where we have used an integration by parts and applied the equality (cf. [24, Lemma IV.1.15] or [25, (3.13)])

$$\int_\alpha^\beta \frac{\log s}{\sqrt{(s-\alpha)(\beta-s)}} ds = 2\pi \log \frac{\sqrt{\alpha}+\sqrt{\beta}}{2}. \tag{4.19}$$

On the other hand, from (4.4) and (4.6) we have

$$\begin{aligned} &\int_a^b \frac{x}{x-b} \arccos \frac{(b+a)x-2}{(b-a)x} dx \\ =& \int_a^b \arccos \frac{(b+a)x-2}{(b-a)x} dx + b \int_a^b \frac{1}{x-b} \arccos \frac{(b+a)x-2}{(b-a)x} dx \\ =& \pi(1-a) - 2\pi b \log(1+a). \end{aligned} \tag{4.20}$$

Thus, we obtain from (4.17), (4.18) and (4.20) that

$$I_{62} = -2\pi \log 2 - \pi(1-a) + 2\pi b \log(1+a). \tag{4.21}$$

Recall that $I_6 = I_{61} + I_{62}$. Hence, coupling (4.16) and (4.21) gives (4.8) immediately. $\square$

Corollary 4.2. *Let $\rho(x)$ be as in (2.32), we have*

$$\int_0^b \rho(x)dx = 1. \tag{4.22}$$

Let $g(z)$ be the g – function defined in (2.33), we have

$$g'(z) = -\log \frac{z(b+a) - 2 + 2\sqrt{(z-a)(z-b)}}{z(b-a)} + \frac{-\log c}{2}. \tag{4.23}$$

Let $l := 2g(b) - v(b)$ be the Lagrange multiplier defined in (2.38), we have

$$l = 2\log\frac{b-a}{4} - 2. \tag{4.24}$$

Proof. From (2.32) we have

$$\int_0^b \rho(x)dx = \int_0^a dx + \frac{1}{\pi}\int_a^b \arccos\frac{x(b+a)-2}{x(b-a)}dx.$$

Applying (4.4) to this gives

$$\int_0^b \rho(x)dx = a + 1 - a = 1,$$

thus proving (4.22).

From (2.32) and (2.33) we obtain

$$\begin{aligned} g'(z) &= \int_0^b \frac{1}{z-x}\rho(x)dx \\ &= \int_0^a \frac{1}{z-x}dx + \frac{1}{\pi}\int_a^b \frac{1}{z-x}\arccos\frac{x(b+a)-2}{x(b-a)}dx. \end{aligned}$$

On account of (4.5) and the equality

$$\int_0^a \frac{1}{z-x}dx = \log\frac{z}{z-a},$$

we have

$$\begin{aligned} g'(z) &= \log\frac{z}{z-a} - \log\frac{z(b+a)-2+2\sqrt{(z-a)(z-b)}}{(z-a)(1+a)(1+b)} \\ &= -\log\frac{z(b+a)-2+2\sqrt{(z-a)(z-b)}}{z(1+a)(1+b)} \\ &= -\log\frac{z(b+a)-2+2\sqrt{(z-a)(z-b)}}{z(b-a)} - \log\frac{b-a}{(1+a)(1+b)}. \end{aligned}$$

Applying (2.31) to the last equation yields (4.23).

Applying (2.32) to (2.33) gives

$$g(b) = \int_0^a \log(b-x)dx + \frac{1}{\pi}\int_a^b \log(b-x)\arccos\frac{x(b+a)-2}{x(b-a)}dx.$$

From (4.8) and the equality

$$\int_0^a \log(b-x)dx = (a-b)\log(b-a) - a + b\log b,$$

we have

$$\begin{aligned} g(b) &= [(a-b)\log(b-a) - a + b\log b] \\ &\quad + [2b\log(1+a) + (1-a)\log(b-a) - 2\log 2 - 1 + a] \\ &= \log\frac{b-a}{4} - 1 + b\log\frac{b(1+a)^2}{b-a}. \end{aligned}$$

Since $v(b) = -b\log c$ by (2.21), we obtain from (2.31) that

$$g(b) - v(b)/2 = \log\frac{b-a}{4} - 1.$$

This proves (4.24). □

4.2 The equilibrium measure of the Meixner polynomials

As mentioned in Remark 2.7, we could solve the equilibrium measure in a standard procedure which contains three steps.

Firstly, we find the Mhaskar-Rakhmanov-Saff numbers a and b by solving the equations (cf. [3, (759)]):

$$\int_a^b \frac{v'(x)}{\sqrt{(x-a)(b-x)}}dx - \int_0^a \frac{2\pi}{\sqrt{(a-x)(b-x)}}dx = 0,$$
$$\int_a^b \frac{xv'(x)}{\sqrt{(x-a)(b-x)}}dx - \int_0^a \frac{2\pi x}{\sqrt{(a-x)(b-x)}}dx = 2\pi,$$

where $v(z) := -z\log c$ is defined in (2.21). A simple calculation gives

$$-\pi\log c - 2\pi \operatorname{arccosh}\frac{b+a}{b-a} = 0, \tag{4.25}$$
$$(-\log c)\cdot(\frac{b+a}{2}\pi) - 2\pi[\frac{b+a}{2}\operatorname{arccosh}\frac{b+a}{b-a} - \sqrt{ab}] = 2\pi, \tag{4.26}$$

where we have used the equalities (by a change of variable $x = \frac{b+a}{2} - \frac{b-a}{2}t$):

$$\int_a^b \frac{1}{\sqrt{(x-a)(b-x)}}dx = \int_{-1}^1 \frac{1}{\sqrt{1-t^2}}dt = \pi,$$
$$\int_0^a \frac{1}{\sqrt{(a-x)(b-x)}}dx = \int_1^{\frac{b+a}{b-a}} \frac{1}{\sqrt{t^2-1}}dt = \operatorname{arccosh}\frac{b+a}{b-a},$$
$$\int_a^b \frac{x}{\sqrt{(x-a)(b-x)}}dx = \int_{-1}^1 \frac{\frac{b+a}{2} - \frac{b-a}{2}t}{\sqrt{1-t^2}}dt = \frac{b+a}{2}\pi,$$
$$\int_0^a \frac{x}{\sqrt{(a-x)(b-x)}}dx = \int_1^{\frac{b+a}{b-a}} \frac{\frac{b+a}{2} - \frac{b-a}{2}t}{\sqrt{t^2-1}}dt = \frac{b+a}{2}\operatorname{arccosh}\frac{b+a}{b-a} - \sqrt{ab}.$$

From (4.25) we have

$$\operatorname{arccosh}\frac{b+a}{b-a} = \frac{-\log c}{2}.$$

Applying this to (4.26) yields $ab = 1$, and thus

$$a = \frac{1-\sqrt{c}}{1+\sqrt{c}}, \qquad b = \frac{1+\sqrt{c}}{1-\sqrt{c}}.$$

This agrees with (2.31).

In the second step, it can be shown that the function $g'(z)$ has following explicit integral representation (cf. [3, (758)]):

$$g'(z) = \int_0^a \frac{\sqrt{(z-a)(z-b)}}{\sqrt{(a-x)(b-x)}}\frac{dx}{x-z} - \int_a^b \frac{\sqrt{(z-a)(z-b)}}{\sqrt{(x-a)(b-x)}}\frac{v'(x)dx}{2\pi(x-z)}. \tag{4.27}$$

To calculate $g'(z)$, we shall use the integral equality

$$\int_a^b \frac{1}{\sqrt{(x-a)(b-x)}} \frac{dx}{x-z} = \frac{-\pi}{\sqrt{(z-a)(z-b)}}, \tag{4.28}$$

Set $\alpha = z-b$ and $\beta = z-a$ in (4.19), it follows from a change of variable $s = z-x$ that

$$\int_a^b \frac{\log(z-x)dx}{\sqrt{(x-a)(b-x)}} = 2\pi \log \frac{\sqrt{z-a}+\sqrt{z-b}}{2}.$$

Differentiate both side of the last equation with respect to z, then (4.28) follows. Form (2.31) and (4.28) we have

$$\int_a^b \frac{\sqrt{(z-a)(z-b)}}{\sqrt{(x-a)(b-x)}} \frac{v'(x)dx}{2\pi(x-z)} = \frac{\log c}{2}. \tag{4.29}$$

Applying this and (4.7) to (4.27) gives

$$g'(z) = -\log \frac{z(b+a)-2+2\sqrt{(z-a)(z-b)}}{z(b-a)} + \frac{-\log c}{2}.$$

This is coincident with (2.34).

Finally, the equilibrium measure $\rho(x)dx$ can be obtained from the equation (cf. [3, (711)]):

$$\rho(x) = \frac{g'_-(x) - g'_+(x)}{2\pi i}. \tag{4.30}$$

Since $ab = 1$, a direct calculation shows that

$$g'_\pm(x) = -\log \frac{-x(b+a)+2+2\sqrt{(a-x)(b-x)}}{x(b-a)} \mp i\pi + \frac{-\log c}{2}$$

for $0 < x < a$, and

$$\begin{aligned} g'_\pm(x) &= -\log \frac{x(b+a)+2 \pm 2i\sqrt{(x-a)(b-x)}}{x(b-a)} + \frac{-\log c}{2} \\ &= \mp i \arccos \frac{x(b+a)-2}{x(b-a)} + \frac{-\log c}{2} \end{aligned}$$

for $a < x < b$. Therefore, we obtain from (4.30) that $\rho(x) = 1$ for $0 < x < a$, and

$$\rho(x) = \frac{1}{\pi} \arccos \frac{x(b+a)-2}{x(b-a)}$$

for $a < x < b$. This agrees with formula (2.32).

4.3 The function $D(z)$

We intend to show that as $n \to \infty$, the function $D(z)$ defined in (2.109) converges uniformly to the constant "1" for z bounded away from the origin. Recall from (2.109) that

$$D(z) = \exp\left\{\frac{1}{2\pi i}\int_0^\infty \left[\frac{\log(1-e^{-2n\pi s - i\pi\beta})}{s+iz} - \frac{\log(1-e^{-2n\pi s+i\pi\beta})}{s-iz}\right] ds\right\}.$$

Given any small $\varepsilon > 0$, we will show that the integral in the brackets is uniformly bounded by $O(1/n)$ for $|z| \geq \varepsilon$. Without loss of generality, we only consider the integral

$$\int_0^\infty \frac{\log(1-e^{-2n\pi s - i\pi\beta})}{s+iz} ds = \int_0^{\varepsilon/2} + \int_{\varepsilon/2}^\infty =: I_1 + I_2.$$

In view of the inequalities

$$\begin{aligned}
|\log(1-re^{i\theta})| &= \left|\log|1-re^{i\theta}| + i\arctan\frac{r\sin\theta}{1-r\cos\theta}\right| \\
&\leq -\log(1-r) + \frac{r|\sin\theta|}{1-r\cos\theta}
\end{aligned}$$

for any $0 < r < 1$ and $\theta \in \mathbb{R}$, we obtain for any $|z| \geq \varepsilon$,

$$\begin{aligned}
I_1 &\leq \int_0^{\varepsilon/2} \frac{|\log(1-e^{-2n\pi s - i\pi\beta})|}{|s+iz|} ds \\
&\leq \frac{1}{\varepsilon/2}\int_0^{\varepsilon/2}\left[-\log(1-e^{-2n\pi s}) + \frac{e^{-2n\pi s}|\sin\pi\beta|}{1-e^{-2n\pi s}\cos\pi\beta}\right] ds.
\end{aligned}$$

Since

$$\int_0^{\varepsilon/2} [-\log(1-e^{-2n\pi s})] ds \leq -\varepsilon/2 \cdot \log(1-e^{-n\pi\varepsilon}) = O(e^{-n\pi\varepsilon}),$$

and

$$\int_0^{\varepsilon/2}\left[\frac{e^{-2n\pi s}|\sin\pi\beta|}{1-e^{-2n\pi s}\cos\pi\beta}\right] ds = \begin{cases} \dfrac{1-e^{-n\pi\varepsilon}}{2n\pi} & : \qquad \cos\pi\beta = 0, \\[2ex] \dfrac{|\sin\beta|}{2n\pi\cos\beta}\log\dfrac{1-e^{-n\pi\varepsilon}\cos\beta}{1-\cos\beta} & : \qquad \cos\pi\beta \neq 0, \end{cases}$$

we have as $n \to \infty$,

$$|I_1| = O(1/n).$$

To estimate the integral I_2, we deform the interval $[\varepsilon/2, \infty)$ to a suitable contour Γ such that $|\zeta + iz| \geq \varepsilon/2$ and $\operatorname{Re}\zeta \geq \varepsilon/2$ for $\zeta \in \Gamma$. We also require the length of $\Gamma \setminus [\varepsilon/2, \infty)$ is less than or equal to $\pi\varepsilon/2$. If $\operatorname{Re}(-iz) \leq \varepsilon/2$ or $|\operatorname{Im}(-iz)| \geq \varepsilon/2$, we choose Γ to be the same as the interval $[\varepsilon/2, \infty)$. Otherwise, when $\operatorname{Re}(-iz) > \varepsilon/2$ and $|\operatorname{Im}(-iz)| < \varepsilon/2$, set

$$\Gamma = [\varepsilon/2, \operatorname{Re}(-iz) - \delta] \cup \gamma \cup [\operatorname{Re}(-iz) + \delta, \infty],$$

where $\delta = \sqrt{(\varepsilon/2)^2 - (\operatorname{Im}(-iz))^2}$ and γ is the curve with two end points $\operatorname{Re}(-iz) \pm \delta$ such that the distance between $(-iz)$ and any point on γ is $\varepsilon/2$. Therefore, γ is the part of the circle

$$U(-iz, \varepsilon/2) := \{w \in \mathbb{C} : |w + iz| = \varepsilon/2\}$$

in the upper (or lower) half plane with respect to $-iz$ is in the lower (or upper) half plane. By Cauchy's theorem we have

$$I_2 = \int_\Gamma \frac{\log(1 - e^{-2n\pi\zeta - i\pi\beta})}{\zeta + iz} d\zeta.$$

By virtue of the inequality

$$|\log(1 - re^{i\theta})| \leq -\log(1 - r) + \frac{r|\sin\theta|}{1 - r\cos\theta} \leq \frac{r}{1-r} + \frac{r}{1-r} = \frac{2r}{1-r}$$

for any $0 < r < 1$ and $\theta \in \mathbb{R}$, we obtain

$$\begin{aligned}
|I_2| \leq & \int_\Gamma \frac{|\log(1 - e^{-2n\pi\zeta - i\pi\beta})|}{|\zeta + iz|} |d\zeta| \\
\leq & \frac{1}{\varepsilon/2} \int_\Gamma \frac{2e^{-2n\pi \operatorname{Re}\zeta}}{1 - e^{-2n\pi \operatorname{Re}\zeta}} |d\zeta| \\
\leq & \frac{4/\varepsilon}{1 - e^{-n\pi\varepsilon}} \int_\Gamma e^{-2n\pi \operatorname{Re}\zeta} |d\zeta| \\
\leq & \frac{4/\varepsilon}{1 - e^{-n\pi\varepsilon}} \left[\int_\gamma e^{-2n\pi \operatorname{Re}\zeta} |d\zeta| + \int_{\varepsilon/2}^\infty e^{-2n\pi \operatorname{Re}\zeta} |d\zeta| \right] \\
\leq & \frac{4/\varepsilon}{1 - e^{-n\pi\varepsilon}} \left[\frac{\varepsilon}{2} e^{-n\pi\varepsilon} + \frac{e^{-n\pi\varepsilon}}{2n\pi} \right] \\
= & O(e^{-n\pi\varepsilon}).
\end{aligned}$$

Coupling the estimates for I_1 and I_2 implies for $|z| \geq \varepsilon$,

$$\int_0^\infty \frac{\log(1 - e^{-2n\pi s - i\pi\beta})}{s + iz} ds = O(1/n).$$

Similarly, we can prove

$$\int_0^\infty \frac{\log(1 - e^{-2n\pi s + i\pi\beta})}{s - iz} ds = O(1/n).$$

Hence, applying the last two estimates to (2.109) yields

$$D(z) = 1 + O(1/n),$$

which holds uniformly for z bounded away from zero.

4.4 The parameter β of the Meixner polynomials

In this section we intend to show that the assumption $1 \leq \beta < 2$ in Theorem 2.21 can be replaced by $\beta > 0$. First, we prove that formula (2.133) in Theorem 2.21 is still true when β is replaced by

$$\beta_- := \beta - 1 \tag{4.31}$$

or

$$\beta_+ := \beta + 1. \tag{4.32}$$

In view of Gauss's contiguous relations for hypergeometric functions [1, (15.2.17) and (15.2.20)], we obtain from (1.1) and (2.1) that

$$\begin{aligned} \pi_n(nz - \beta_-/2) = & \frac{nz + \beta_-/2}{n + \beta_-} \pi_n(nz_1 - \beta/2) \\ & - \frac{nz - \beta_-/2}{n + \beta_-} \pi_n(nz_2 - \beta/2), \end{aligned} \tag{4.33}$$

and

$$\pi_n(nz - \beta_+/2) = \frac{1}{1 - c} \pi_n(nz_2 - \beta/2) - \frac{c}{1 - c} \pi_n(nz_1 - \beta/2), \tag{4.34}$$

where

$$z_1 := z + 1/(2n), \qquad z_2 := z - 1/(2n). \tag{4.35}$$

If z belongs to the region $\Omega^4 \cup \Omega^\infty$, so do z_1 and z_2. Therefore, applying (2.133) to the two polynomials on the right-hand side of (4.33) gives

$$\begin{aligned}\pi_n(nz-\beta_-/2) = {} & zn^n e^{ng(z_1)} \frac{z_1^{(1-\beta)/2}(\frac{\sqrt{z_1-a}+\sqrt{z_1-b}}{2})^\beta}{(z_1-a)^{1/4}(z_1-b)^{1/4}}\left[1+O(\frac{1}{n})\right] \\ & - zn^n e^{ng(z_2)} \frac{z_2^{(1-\beta)/2}(\frac{\sqrt{z_2-a}+\sqrt{z_2-b}}{2})^\beta}{(z_2-a)^{1/4}(z_2-b)^{1/4}}\left[1+O(\frac{1}{n})\right].\end{aligned}$$

On account of (4.31) and (4.35), we obtain

$$\begin{aligned}\pi_n(nz-\beta_-/2) = {} & n^n e^{ng(z)} \frac{z^{(1-\beta_-)/2}(\frac{\sqrt{z-a}+\sqrt{z-b}}{2})^{\beta_-}}{(z-a)^{1/4}(z-b)^{1/4}} \frac{\sqrt{z}(\sqrt{z-a}+\sqrt{z-b})}{2} \\ & \times[e^{ng(z_1)-ng(z)} - e^{ng(z_2)-ng(z)}]\left[1+O(1/n)\right].\end{aligned} \tag{4.36}$$

Also, we have from (4.35)

$$ng(z_1) - ng(z) = g'(z)/2 + O(1/n), \qquad ng(z_2) - ng(z) = -g'(z)/2 + O(1/n).$$

Thus, it follows from (2.34) that

$$e^{ng(z_1)-ng(z)} = \lambda[1+O(1/n)], \qquad e^{ng(z_2)-ng(z)} = \lambda^{-1}[1+O(1/n)], \tag{4.37}$$

where

$$\lambda := \left[\frac{z(b-a)/\sqrt{c}}{z(b+a)-2+2\sqrt{(z-a)(z-b)}}\right]. \tag{4.38}$$

In view of (4.11), we obtain

$$\lambda = \frac{z+1+\sqrt{(z-a)(z-b)}}{\sqrt{z}(\sqrt{z-a}+\sqrt{z-b})}\left[\frac{(b-a)/\sqrt{c}}{(1+a)(1+b)}\right]^{1/2}.$$

Since $(b-a)/\sqrt{c} = (1+a)(1+b)$ by (2.31), it is easily seen that

$$\lambda = \frac{z+1+\sqrt{(z-a)(z-b)}}{\sqrt{z}(\sqrt{z-a}+\sqrt{z-b})},$$

and thus

$$\lambda - \lambda^{-1} = \frac{2}{\sqrt{z}(\sqrt{z-a}+\sqrt{z-b})}.$$

Applying this and (4.37) to (4.36) yields

$$\pi_n(nz - \beta_-/2) = n^n e^{ng(z)} \frac{z^{(1-\beta_-)/2}(\frac{\sqrt{z-a}+\sqrt{z-b}}{2})^{\beta_-}}{(z-a)^{1/4}(z-b)^{1/4}} \left[1 + O(\frac{1}{n})\right]. \tag{4.39}$$

On the other hand, by applying (2.133) to the two polynomials on the right-hand side of (4.34), we obtain

$$\begin{aligned}\pi_n(nz - \beta_+/2) = {} & \frac{1}{1-c} n^n e^{ng(z_2)} \frac{z_2^{(1-\beta)/2}(\frac{\sqrt{z_2-a}+\sqrt{z_2-b}}{2})^{\beta}}{(z_2-a)^{1/4}(z_2-b)^{1/4}} \left[1 + O(\frac{1}{n})\right] \\ & - \frac{c}{1-c} n^n e^{ng(z_1)} \frac{z_1^{(1-\beta)/2}(\frac{\sqrt{z_1-a}+\sqrt{z_1-b}}{2})^{\beta}}{(z_1-a)^{1/4}(z_1-b)^{1/4}} \left[1 + O(\frac{1}{n})\right].\end{aligned}$$

On account of (4.32), (4.35) and (4.37), we have

$$\begin{aligned}\pi_n(nz - \beta_+/2) = {} & n^n e^{ng(z)} \frac{z^{(1-\beta_+)/2}(\frac{\sqrt{z-a}+\sqrt{z-b}}{2})^{\beta_+}}{(z-a)^{1/4}(z-b)^{1/4}} \frac{2\sqrt{z}/(1-c)}{\sqrt{z-a}+\sqrt{z-b}} \\ & \times[\lambda^{-1} - c\lambda]\left[1 + O(1/n)\right].\end{aligned} \tag{4.40}$$

Since $(b+a) - \sqrt{c}(b-a) = 2$ by (2.31), we have from (4.38)

$$\lambda^{-1} - c\lambda = \frac{1 - c\lambda^2}{\lambda} = \frac{2(z - 1 + \sqrt{(z-a)(z-b)})}{[z(b+a) - 2 + 2\sqrt{(z-a)(z-b)}]^{1/2}[z(b-a)/\sqrt{c}]^{1/2}}.$$

In view of (4.11), we obtain

$$\lambda^{-1} - c\lambda = \frac{2(z - 1 + \sqrt{(z-a)(z-b)})(z + 1 + \sqrt{(z-a)(z-b)})}{z(\sqrt{z-a}+\sqrt{z-b})[(1+a)(1+b)]^{1/2}[z(b-a)/\sqrt{c}]^{1/2}}.$$

Since

$$(z - 1 + \sqrt{(z-a)(z-b)})(z + 1 + \sqrt{(z-a)(z-b)}) = z(\sqrt{z-a}+\sqrt{z-b})^2$$

and

$$[(1+a)(1+b)]^{1/2}[(b-a)/\sqrt{c}]^{1/2} = 4/(1-c)$$

by (2.31), we have

$$\lambda^{-1} - c\lambda = \frac{\sqrt{z-a}+\sqrt{z-b}}{2\sqrt{z}/(1-c)}.$$

Applying this to (4.40) gives

$$\pi_n(nz - \beta_+/2) = n^n e^{ng(z)} \frac{z^{(1-\beta_+)/2}(\frac{\sqrt{z-a}+\sqrt{z-b}}{2})^{\beta_+}}{(z-a)^{1/4}(z-b)^{1/4}} \left[1 + O(\frac{1}{n})\right]. \tag{4.41}$$

In view of (4.39) and (4.41), it can be shown by induction that formula (2.133) in Theorem 2.21 is valid for all $\beta > 0$. Similarly, we could prove that formulas (2.134)-(2.140) in Theorem 2.21 are also valid for $\beta > 0$.

4.5 The asymptotic formulas for the Meixner polynomials

We first provide some numerical computations by using our results in Theorem 2.21. Choosing $c = 0.5$, it is easily seen from (2.31) that $a \approx 0.17157$ and $b \approx 5.82843$. We also fix $\beta = 1.5$. Since the polynomial degree n should be reasonably large, we set $n = 100$. The approximate values of $\pi_n(nz-\beta/2)$ are obtained by using the asymptotic formulas given in Theorem 2.21. We use formula (2.133) for $z = -1$ and $z = 100$, formula (2.135)-(2.136) for $z = \pm 0.001$, formula (2.134) for $z = 0.05$, formula (2.137) for $z = 0.171$ and $z = 0.172$, formula (2.139) or (2.140) for $z = 2$, and formula (2.138) for $z = 5.828$ and $z = 5.829$. The true values of $\pi_n(nz - \beta/2)$ can be obtained from (1.1) and (2.1). The numerical results are presented in Table 4.1.

Now, we compare our formulas in Theorem 2.21 with those given in [16] and [17]. We shall introduce two notations. Let

$$\alpha := z - \beta/(2n) \tag{4.42}$$

and

$$m_n(n\alpha; \beta, c) := (1 - 1/c)^n \pi_n(nz - \beta/2). \tag{4.43}$$

Two different asymptotic formulas for $m_n(n\alpha; \beta, c)$ are given in [16, (6.9) and (6.27)]; both in terms of parabolic cylinder functions. To study the large and

	True value	Approximate value
$z=-1$	1.99529×10^{233}	1.99473×10^{233}
$z=-0.001$	8.36624×10^{187}	8.35137×10^{187}
$z=0.001$	3.07930×10^{187}	3.07272×10^{187}
$z=0.05$	-2.51701×10^{180}	-2.51507×10^{180}
$z=0.171$	-9.12697×10^{174}	-9.12530×10^{174}
$z=0.172$	-1.22035×10^{175}	-1.22003×10^{175}
$z=2$	-4.71541×10^{201}	-4.70772×10^{201}
$z=5.828$	2.78146×10^{259}	2.78231×10^{259}
$z=5.829$	2.86933×10^{259}	2.87018×10^{259}
$z=100$	2.16586×10^{399}	2.16586×10^{399}

Table 4.1 The true values and approximate values of $\pi_n(nz-\beta/2)$ for $c=0.5, \beta=1.5$ and $n=100$. Note that $a\approx 0.17157$ and $b\approx 5.82843$.

small zeros of the Meixner polynomials, these two formulas are transformed to (2.35) and (4.19) in [17]. Here, we intend to show the equivalence between our equation (2.138) and (2.35) in [17], and also the equivalence between our equation (2.134) and (4.19) in [17].

In view of [17, (2.34)], we rewrite the formula [17, (2.35)] as

$$\begin{aligned} m_n(n\alpha;\beta,c) \sim (-1)^n\sqrt{2\pi}n^{n+1/6}e^{n(\gamma+\eta^2/4-1/2)} \\ \times c^{-1/6}(1+\sqrt{c})^{2/3-\beta}\,\mathrm{Ai}(n^{2/3}(\eta-2)), \end{aligned} \tag{4.44}$$

where γ is a constant and η is a function of α. The constant γ and the function η could be solved from the following two equations (cf. [16, (3.12)-(3.13)]):

$$\alpha\log\frac{1-w_+/c}{1-w_+}-\log(-w_+)=-\log u_-+\eta u_--u_-^2/2+\gamma, \tag{4.45}$$

$$\alpha\log\frac{1-w_-/c}{1-w_-}-\log(-w_-)=-\log u_++\eta u_+-u_+^2/2+\gamma. \tag{4.46}$$

The saddle points $w_\pm$ and $u_\pm$ are given by (cf. [16, (2.5) and (3.8)])

$$\begin{aligned}
w_\pm &= \frac{1+c+\alpha c-\alpha \pm \sqrt{(1+c+\alpha c-\alpha)^2-4c}}{2}, \\
u_\pm &= \eta/2 \pm \sqrt{\eta^2/4-1};
\end{aligned}$$

see also [17, (2.4)-(2.5)]. Adding (4.45) to (4.46) gives

$$\eta^2/4+\gamma+1/2 = -\frac{\alpha+1}{2}\log c. \tag{4.47}$$

Subtracting (4.45) from (4.46) yields

$$\begin{aligned}
&(\eta/2)\sqrt{\eta^2/4-1}+\log(\eta/2-\sqrt{\eta^2/4-1}) \\
&= \frac{\alpha}{2}\log\frac{(1-w_-/c)(1-w_+)}{(1-w_+/c)(1-w_-)}+\frac{1}{2}\log\frac{w_+}{w_-}.
\end{aligned} \tag{4.48}$$

From the definition of ϕ-function in (2.37), we have $\phi(b)=0$ and

$$\begin{aligned}
\phi'(\alpha) &= \log\frac{\alpha(b+a)-2+2\sqrt{(\alpha-a)(\alpha-b)}}{\alpha(b-a)} \\
&= \frac{1}{2}\log\frac{(1-w_-/c)(1-w_+)}{(1-w_+/c)(1-w_-)},
\end{aligned}$$

where we have used

$$a = \frac{1-\sqrt{c}}{1+\sqrt{c}} \qquad \text{and} \qquad b = \frac{1+\sqrt{c}}{1-\sqrt{c}};$$

see (2.31). Therefore, we obtain from (4.48)

$$\phi(\alpha) = (\eta/2)\sqrt{\eta^2/4-1}+\log(\eta/2-\sqrt{\eta^2/4-1}).$$

Recall from [17, p.284] that $\eta-2=O(n^{-2/3})$. We then have

$$\phi(\alpha)\sim\frac{2}{3}(\eta-2)^{3/2}.$$

Applying this to (2.118) yields

$$F(\alpha)\sim n^{2/3}(\eta-2). \tag{4.49}$$

A combination of (4.42)-(4.44), (4.47) and (4.49) gives

$$\pi_n(nz-\beta/2) \sim \sqrt{2\pi} n^{n+1/6} e^{-n} c^{-nz/2+n/2+\beta/4-1/6} \times (1-c)^{-n}(1+\sqrt{c})^{2/3-\beta}\,\mathrm{Ai}(F(z)). \tag{4.50}$$

Applying (2.45) to (2.118) implies

$$\frac{F(z)}{z-b} \sim \left(\frac{2n}{b\sqrt{b-a}}\right)^{2/3} = n^{2/3}c^{-1/6}(1-c)(1+\sqrt{c})^{-4/3}.$$

Therefore, we have

$$\frac{(\frac{\sqrt{z-a}+\sqrt{z-b}}{2})^\beta + (\frac{\sqrt{z-a}-\sqrt{z-b}}{2})^\beta}{z^{(\beta-1)/2}(z-a)^{1/4}(z-b)^{1/4}F(z)^{-1/4}} \sim \sqrt{2} n^{1/6} c^{\beta/4-1/6}(1+\sqrt{c})^{2/3-\beta}. \tag{4.51}$$

Moreover, it is easy to see that

$$\frac{(\frac{\sqrt{z-a}+\sqrt{z-b}}{2})^\beta - (\frac{\sqrt{z-a}-\sqrt{z-b}}{2})^\beta}{z^{(\beta-1)/2}(z-a)^{1/4}(z-b)^{1/4}F^{1/4}} = O(n^{-1/6}). \tag{4.52}$$

From (2.21) and (2.38), we obtain

$$e^{nv/2+nl/2} = e^{-n}c^{-nz/2+n/2}(1-c)^{-n}. \tag{4.53}$$

Hence, we can derive (4.50) again by applying (4.51)-(4.53) to (2.138). This establishes the equivalence between (2.138) and [17, (2.35)].

Applying [17, (3.4)] and [17, (3.11)-(3.12)] to [17, (4.19)], we have

$$\begin{aligned} m_n(n\alpha;\beta,c) = {} & \frac{2n^{n\alpha}n!\alpha^{n\alpha+1/2}}{\Gamma(n\alpha+1)} \exp\left\{n(\gamma+\eta^2/4-\alpha/2)\right\} \\ & \times \exp\left\{n\left[-\alpha\log(-\eta/2+\sqrt{\eta^2/4-\alpha}) - (\eta/2)\sqrt{\eta^2/4-\alpha}\right]\right\} \\ & \times \frac{-h(u_-)}{(\eta^2-4\alpha)^{1/4}\sqrt{-u_-}} \\ & \times \left\{\sin n\pi\alpha\left[1+O(\frac{1}{n})\right] + O(\alpha^{-1/2}e^{-2\varepsilon_0 n})\right\}. \end{aligned} \tag{4.54}$$

Here again, γ is a constant and η is a function of α. and they can be solved from the two equations (cf. [16, (3.23)-(3.24)]):

$$\alpha\log\frac{1-w_+/c}{1-w_+} - \log w_+ = -\alpha\log u_+ + \eta u_+ - u_+^2/2 + \gamma, \tag{4.55}$$

$$\alpha\log\frac{1-w_-/c}{1-w_-} - \log w_- = -\alpha\log u_- + \eta u_- - u_-^2/2 + \gamma. \tag{4.56}$$

The saddle points $w_\pm$ and $u_\pm$ are given by (cf. [16, (2.5) and (3.22)])

$$\begin{aligned} w_\pm &= \frac{1+c+\alpha c-\alpha \pm \sqrt{(1+c+\alpha c-\alpha)^2-4c}}{2}, \\ u_\pm &= \eta/2 \pm \sqrt{\eta^2/4-\alpha}; \end{aligned}$$

see also [17, (3.3)-(3.4)]. Adding (4.55) to (4.56) yields

$$\eta^2/4+\gamma = -\frac{\alpha+1}{2}\log c - \alpha/2 + \frac{\alpha}{2}\log\alpha. \tag{4.57}$$

Subtracting (4.55) from (4.56) gives

$$\begin{aligned} (-\eta/2)\sqrt{\eta^2/4-\alpha} - \alpha\log(-\eta/2+\sqrt{\eta^2/4-\alpha}) + (\alpha/2)\log\alpha \\ = \frac{\alpha}{2}\log\frac{(w_-/c-1)(1-w_+)}{(w_+/c-1)(1-w_-)} + \frac{1}{2}\log\frac{w_+}{w_-}. \end{aligned} \tag{4.58}$$

Recall from (2.31) that

$$a = \frac{1-\sqrt{c}}{1+\sqrt{c}} \qquad \text{and} \qquad b = \frac{1+\sqrt{c}}{1-\sqrt{c}}.$$

Therefore, from the definition of $\widetilde{\phi}$-function in (2.39) we have $\widetilde{\phi}(a) = 0$ and

$$\begin{aligned} \widetilde{\phi}'(\alpha) &= \log\frac{-\alpha(b+a)+2+2\sqrt{(a-\alpha)(b-\alpha)}}{\alpha(b-a)} \\ &= -\frac{1}{2}\log\frac{(w_-/c-1)(1-w_+)}{(w_+/c-1)(1-w_-)}. \end{aligned}$$

On account of (4.58), we otain

$$\begin{aligned} -\widetilde{\phi}(\alpha) = &(-\eta/2)\sqrt{\eta^2/4-\alpha} - \alpha\log(-\eta/2+\sqrt{\eta^2/4-\alpha}) \\ &+(\alpha/2)\log\alpha. \end{aligned} \tag{4.59}$$

Furthermore, a direct calculation shows that

$$\frac{-h(u_-)}{(\eta^2-4\alpha)^{1/4}\sqrt{-u_-}} = -\sqrt{\alpha}[(a-\alpha)(b-\alpha)]^{-1/4}(1-w_-)^{-\beta}. \tag{4.60}$$

In view of (4.42) and the equality

$$\frac{-\alpha(b+a)+2+2\sqrt{(a-\alpha)(b-\alpha)}}{b-a} = c^{-1/2}(1-w_-)^2\left(\frac{\sqrt{b-\alpha}+\sqrt{a-\alpha}}{2}\right)^2$$

we have

$$
\begin{aligned}
&\exp\{-n\widetilde{\phi}(\alpha)+n\widetilde{\phi}(z)\}\\
=&\exp\{(\beta/2)\widetilde{\phi}'(\alpha)+O(1/n)\}\\
=&\left[\frac{-\alpha(b+a)+2+2\sqrt{(a-\alpha)(b-\alpha)}}{\alpha(b-a)}\right]^{\beta/2}\left[1+O(\frac{1}{n})\right]\\
=&\alpha^{-\beta/2}c^{-\beta/4}(1-w_-)^{\beta}\left(\frac{\sqrt{b-\alpha}+\sqrt{a-\alpha}}{2}\right)^{\beta}\left[1+O(\frac{1}{n})\right].
\end{aligned}
\tag{4.61}
$$

It can be shown by Stirling's formula that

$$
\frac{2n^{n\alpha}n!}{\Gamma(n\alpha+1)}\alpha^{n\alpha+1/2}e^{-n\alpha/2}=2n^{n}e^{-n+n\alpha/2}\left[1+O(\frac{1}{n})\right].
\tag{4.62}
$$

Applying (4.57) and (4.59)-(4.62) to (4.54) gives

$$
\begin{aligned}
m_n(n\alpha;\beta,c)=&-2n^{n}e^{-n}c^{-n\alpha/2-\beta/4-n/2}e^{-n\widetilde{\phi}(z)}\frac{\alpha^{(1-\beta)/2}(\frac{\sqrt{b-\alpha}+\sqrt{a-\alpha}}{2})^{\beta}}{[(a-\alpha)(b-\alpha)]^{1/4}}\\
&\times\left\{\sin n\pi\alpha\left[1+O(\frac{1}{n})\right]+O(\alpha^{-1/2}e^{-2\varepsilon_0 n})\right\},
\end{aligned}
\tag{4.63}
$$

which is exactly the same as (2.134) in view of (4.42), (4.43) and (4.53).

4.6 The asymptotic formulas for some q-orthogonal polynomials

We first compare our formulas for three classes of q-orthogonal polynomials in Section 3.3 with those given in [15]. We only take the q^{-1}-Hermite polynomials for example. Similar arguments go for the Stieltjes-Wigert polynomials and the q-Laguerre polynomials. Recall that the scale for the q^{-1}-Hermite polynomials is given by

$$
z=\sinh\xi:=(q^{-nt}u-q^{nt}u^{-1})/2
$$

with $u\neq 0$ and $t\geq 0$. For $t\geq 1/2$ and $|u|\geq 1/R$, where $R>0$ is any fixed large number, from (3.21) we have

$$
\begin{aligned}
h_n(\sinh\xi|q)=&u^{n}q^{-n^2t}\big[A_q(u^{-2}q^{n(2t-1)})\\
&+A_{q,n}(u^{-2}q^{n(2t-1)})-A_q(u^{-2}q^{n(2t-1)})+r_n(t,u)\big].
\end{aligned}
\tag{4.64}
$$

Given any small $\delta > 0$, we intend to show

$$\left|A_{q,n}(u^{-2}q^{n(2t-1)}) - A_q(u^{-2}q^{n(2t-1)}) + r_n(t,u)\right| = O(q^{n(1-3\delta)}).$$

On the one hand, since $t \geq 1/2$ and $|u| \geq 1/R$, we have $|u^{-2}q^{n(2t-1)}| \leq R^2$. From the definition of the q-Airy function (1.8) and the q-Airy polynomial (1.10) we obtain

$$\begin{aligned}\left|A_{q,n}(u^{-2}q^{n(2t-1)}) - A_q(u^{-2}q^{n(2t-1)})\right| &\leq \sum_{k=n+1}^{\infty} \frac{q^{k^2}}{(q;q)_k} \left|u^{-2}q^{n(2t-1)}\right|^k \\ &\leq \sum_{k=n}^{\infty} \frac{q^{k^2}}{(q;q)_\infty} R^{2k}.\end{aligned}$$

In terms of the half q-Theta function (3.5), we have

$$\sum_{k=n}^{\infty} \frac{q^{k^2}}{(q;q)_\infty} R^{2k} = \sum_{l=0}^{\infty} \frac{q^{(n+l)^2}}{(q;q)_\infty} R^{2n+2l} = \frac{q^{n^2}R^{2n}}{(q;q)_\infty} \Theta_q^+(q^{2n}R^2) = O(q^{n^2(1-\delta)}),$$

and thus

$$\left|A_{q,n}(u^{-2}q^{n(2t-1)}) - A_q(u^{-2}q^{n(2t-1)})\right| = O(q^{n^2(1-\delta)}). \tag{4.65}$$

On the other hand, since $|u|^{-2}q^{n(2t-1)} \leq R^2$, we have from (3.22) that

$$\begin{aligned}|r_n(t,u)| &\leq \frac{q^{n(1-3\delta)}}{1-q} A_{q,n}(-|u|^{-2}q^{n(2t-1)}) + \frac{q^{3n^2\delta^2-2n\delta}R^{2(\lfloor 3n\delta \rfloor+1)}}{(q;q)_\infty} \Theta_q^+(q^{4n\delta}R^2) \\ &\leq \frac{q^{n(1-3\delta)}}{1-q} A_{q,n}(-R^2) + O(q^{3n^2\delta^2(1-\delta)}) \\ &= O(q^{n(1-3\delta)}).\end{aligned} \tag{4.66}$$

Applying (4.65) and (4.66) to (4.64) yields

$$h_n(\sinh\xi|q) = u^n q^{-n^2 t}\left[A_q(u^{-2}q^{n(2t-1)}) + O(q^{n(1-3\delta)})\right]$$

for any small $\delta > 0$. This improves the formula in [15, Theorem 2.1], where their error estimate is $O(q^{n/2})$. Furthermore, when $0 \leq t < 1/2$, we give a single formula (3.23), whereas it takes two formulas in [15] to cover this case, one when t is rational and the other when t is rational. Here, probably it should also be pointed out that the reason why the error estimate in [15, Theorem 2.1] is only $O(n^{-1}\log n)$ when t is irrational is because of the fact that

$$\Theta_q(q^{1/n}) - \Theta_q(1) = O(n^{-1}\log n).$$

In the second part of this section, we intend to show that formula (3.21), together with the error estimate (3.22), can be reduced to formula (3.23) in the case when $1/2-\delta < t < 1/2$ and $1/R \le |u| \le R$, where the constants $\delta \in (0, 1/4)$ and $R > 0$ are fixed. It follows from (3.21) and (3.22) that

$$h_n(\sinh\xi|q) = u^n q^{-n^2 t}\left[A_{q,n}(u^{-2}q^{n(2t-1)}) + r_n(t,u)\right], \tag{4.67}$$

where

$$\begin{aligned} |r_n(t,u)| \le & \frac{q^{n(1-3\delta)}}{1-q} A_{q,n}(-|u|^{-2}q^{n(2t-1)}) \\ & + \frac{q^{3n^2\delta^2-2n\delta}R^{2(\lfloor 3n\delta\rfloor+1)}}{(q;q)_\infty}\Theta_q^+(q^{4n\delta}R^2). \end{aligned} \tag{4.68}$$

Set $m := \lfloor n(1/2-t)\rfloor$. Since $1/R \le |u| \le R$ and $0 < 1-2t < 2\delta < 2$, the conditions of (3.8) in Proposition 3.4 are satisfied with t replaced by $1-2t$. Thus we obtain

$$\begin{aligned} A_{q,n}(u^{-2}q^{-n(1-2t)}) = & \frac{(-u^{-2}q^{-n(1-2t)})^m q^{m^2}}{(q;q)_\infty} \\ & \times\left[\Theta_q(-u^{-2}q^{2m-n(1-2t)}) + O(q^{m(1-\delta)})\right], \end{aligned} \tag{4.69}$$

and

$$\begin{aligned} A_{q,n}(-|u|^{-2}q^{-n(1-2t)}) = & \frac{(|u|^{-2}q^{-n(1-2t)})^m q^{m^2}}{(q;q)_\infty} \\ & \times\left[\Theta_q(|u|^{-2}q^{2m-n(1-2t)}) + O(q^{m(1-\delta)})\right]. \end{aligned} \tag{4.70}$$

Applying (4.69) to (4.67) gives

$$\begin{aligned} h_n(\sinh\xi|q) = & \frac{(-1)^m u^{n-2m} q^{-n^2t-m[n(1-2t)-m]}}{(q;q)_\infty}\Bigg[\Theta_q(-u^{-2}q^{2m-n(1-2t)}) \\ & + O(q^{m(1-\delta)}) + \frac{(q;q)_\infty r_n(t,u)}{(-u^{-2}q^{-n(1-2t)})^m q^{m^2}}\Bigg]. \end{aligned} \tag{4.71}$$

Similarly, applying (4.70) to (4.68) yields

$$\left|\frac{(q;q)_\infty r_n(t,u)}{(-u^{-2}q^{-n(1-2t)})^m q^{m^2}}\right| \le \frac{q^{n(1-3\delta)}}{1-q}\left[\Theta_q(|u|^{-2}q^{2m-n(1-2t)}) + O(q^{m(1-\delta)})\right] + \frac{q^{3n^2\delta^2-2n\delta}R^{2(\lfloor 3n\delta\rfloor+1)}}{(|u|^{-2}q^{-n(1-2t)})^m q^{m^2}}\Theta_q^+(q^{4n\delta}R^2). \tag{4.72}$$

Since $|u| \le R$ and $1/2-\delta < t < 1/2$ by assumption, we have

$$m := \lfloor n(1/2-t)\rfloor \le \lfloor n\delta\rfloor \le n\delta,$$

and thus

$$\frac{q^{3n^2\delta^2-2n\delta}R^{2(\lfloor 3n\delta\rfloor+1)}}{(|u|^{-2}q^{-n(1-2t)})^m q^{m^2}} \le q^{3n^2\delta^2-2n\delta-m^2}R^{2(\lfloor 3n\delta\rfloor+1+m)} = O(q^{2n^2\delta^2(1-\delta)}).$$

Applying this to (4.72) gives

$$\left|\frac{(q;q)_\infty r_n(t,u)}{(-u^{-2}q^{-n(1-2t)})^m q^{m^2}}\right| = O(q^{n(1-3\delta)}) + O(q^{n(1-3\delta)+m(1-\delta)}) + O(q^{2n^2\delta^2(1-\delta)}).$$

Since $\delta \in (0,1/4)$, we obtain

$$m(1-\delta) < m < n\delta < n(1-3\delta),$$

and thus

$$\left|\frac{(q;q)_\infty r_n(t,u)}{(-u^{-2}q^{-n(1-2t)})^m q^{m^2}}\right| = O(q^{m(1-\delta)}). \tag{4.73}$$

Finally, coupling (4.71) and (4.73) yields

$$S_n(z;q) = \frac{(-u)^{n-m}q^{n^2(1-t)-m[n(1-2t)-m]}}{(q;q)_n(q;q)_\infty}\left[\Theta_q(-u^{-2}q^{2m-n(1-2t)}) + O(q^{m(1-\delta)})\right],$$

which agrees with (3.23). Similar arguments can be given to the results for the Stieltjes-Wigert polynomials and the q-Laguerre polynomials in Theorem 3.6 and Theorem 3.7.

Finally, recall the scale $z := q^{-nt}u$ for the Stieltjes-Wigert polynomials and the q-Laguerre polynomials. Note that we only consider the case $t \ge 1$ in Theorem

3.6. Actually, when $t \leq 1$, similar results can be obtained by the symmetry relation of the Stieltjes-Wigert polynomials:

$$S_n(q^{-nt}u;q) = (-u)^n q^{n^2(1-t)} S_n(q^{-n(2-t);q}u^{-1}).$$

For the q-Laguerre polynomials, there is no such kind of relation formula. However, if $0 < t \leq 1$, we can apply Theorem 3.3 to (3.20) with

$$l = 1 - t/2, \quad m = \lfloor nl \rfloor, \quad g_n(k) = k^2 - 2nlk + m(2nl - m),$$
$$f_n(k) = (q^{\alpha+1+n-k};q)_k (q^{n-k+1};q)_k (q^{k+1};q)_\infty, \quad z = -u^{-1}q^{-\alpha}, \quad c_0 = 1, \quad M = 1,$$
$$b_n = 2(m - nl), \quad L = 2, \quad A_\delta = \delta^2(1-\delta), \quad \eta_n(\delta) = 3q^{n(1-l-\delta)}/(1-q).$$

Thus, we obtain

$$L_n^\alpha(z;q) = \frac{(-uq^\alpha)^{n-m} q^{n^2(1-t)-m[n(2-t)-m]}}{(q;q)_n (q;q)_\infty} \times \left[\Theta_q(-u^{-1}q^{2m-n(2-t)-\alpha}) + O(q^{n(1-l-\delta)})\right], \tag{4.74}$$

where $l := 1 - t/2$, $m := \lfloor nl \rfloor$ and $\delta > 0$ is any small number. The asymptotic formula holds uniformly for $u \in T_R := \{z \in \mathbb{C} : R^{-1} \leq |z| \leq R\}$, where $R > 0$ is any large real number.

Note that when $0 < t \leq 1$, we could also apply Theorem 3.3 to the sum (1.7)

$$L_n^\alpha(z;q) = \sum_{k=0}^{n} \frac{(q^{\alpha+k+1};q)_{n-k}}{(q;q)_k (q;q)_{n-k}} q^{k^2+\alpha k} (-q^{-nt}u)^k$$

with

$$\tilde{l} = t/2, \quad \tilde{m} = \lfloor n\tilde{l} \rfloor, \quad g_n(k) = k^2 - 2n\tilde{l}k + \tilde{m}(2n\tilde{l} - \tilde{m}),$$
$$f_n(k) = (q^{\alpha+k+1};q)_{n-k} (q^{n-k+1};q)_k (q^{k+1};q)_\infty, \quad z = -uq^\alpha, \quad c_0 = 1, \quad M = 1,$$
$$b_n = 2(\tilde{m} - n\tilde{l}), \quad L = 2, \quad A_\delta = \delta^2(1-\delta), \quad \eta_n(\delta) = 3q^{n(\tilde{l}-\delta)}/(1-q).$$

Thus, we obtain

$$L_n^\alpha(z;q) = \frac{(-uq^\alpha)^{\tilde{m}} q^{-\tilde{m}[nt-\tilde{m}]}}{(q;q)_n (q;q)_\infty} \left[\Theta_q(-uq^{2\tilde{m}-nt+\alpha}) + O(q^{n(\tilde{l}-\delta)})\right], \tag{4.75}$$

where $\tilde{l} := t/2$, $\tilde{m} := \lfloor n\tilde{l} \rfloor$ and $\delta > 0$ is any small number. The asymptotic formula holds uniformly for $u \in T_R := \{z \in \mathbb{C} : R^{-1} \leq |z| \leq R\}$, where $R > 0$ is any large real number. By (1.9) we have

$$\Theta_q(z) = \Theta_q(1/z)$$

for any $z \in \mathbb{C}$. Therefore, it follows from $m + \tilde{m} = n$ and $l + \tilde{l} = 1$ that the asymptotic formulas (4.74) and (4.75) are exactly the same.

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