UNIFORM ASYMPTOTICS OF THE MEIXNER POLYNOMIALS AND SOME $q$-ORTHOGONAL POLYNOMIALS

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Abstract

In this thesis, we study the uniform asymptotic behavior of the Meixner polynomials and some $q$-orthogonal polynomials as the polynomial degree $n$ tends to infinity.

Using the steepest descent method of Deift-Zhou, we derive uniform asymptotic formulas for the Meixner polynomials. These include an asymptotic formula in a neighborhood of the origin, a result which as far as we are aware has not yet been obtained previously. This particular formula involves a special function, which is the uniformly bounded solution to a scalar Riemann-Hilbert problem, and which is asymptotically (as $n \to \infty$) equal to the constant “1” except at the origin. Numerical computation by using our formulas, and comparison with earlier results, are also given.

With some modifications of Laplace’s approximation, we obtain uniform asymptotic formulas for the Stieltjes-Wigert polynomial, the $q^{-1}$-Hermite polynomial and the $q$-Laguerre polynomial. In these formulas, the $q$-Airy polynomial, defined by truncating the $q$-Airy function, plays a significant role. While the standard Airy function, used frequently in the uniform asymptotic formulas for classical orthogonal polynomials, behaves like the exponential function on one side and the trigonometric functions on the other side of an extreme zero, the $q$-Airy polynomial behaves like the $q$-Airy function on one side and the $q$-Theta function on the other side. The last two special functions are involved in the local asymptotic formulas of the $q$-orthogonal polynomials. It seems therefore reasonable to expect that the $q$-Airy polynomial will play an important role in the asymptotic theory of the $q$-orthogonal polynomials.
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