ACCURATE HIGHER-ORDER ANALYTICAL APPROXIMATIONS TO NONLINEAR OSCILLATION SYSTEMS

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Accurate Higher-order Analytical Approximations to Nonlinear Oscillation Systems
非線性振蕩系統的精確高階分析近似解方法

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The main focus of this dissertation is on the treatment of certain nonlinear oscillation systems by quantitative analysis. The quest for accurate solutions to nonlinear oscillations problems in engineering, applied mathematics, and the physical sciences has led to the development of many analytical or semi-analytical techniques for solving nonlinear differential equations. The concentration on analytical approximations of nonlinear oscillating systems has come about because exact solutions to many nonlinear equations are unavailable, and because numerical integration methods cannot provide an overall view of the nature of the systems in response to changes in parameters that affect nonlinearity. Furthermore, even where an exact solution to a nonlinear equation is available, the resulting expression is presented in terms of implicit functions.

In this research the linearized harmonic balance (LHB) and Newton harmonic balance (NHB) solution methods are presented and elaborated. Both are based on the fundamental idea of the classical harmonic balance (HB) method, which is a convenient procedure for determining analytical approximations of nonlinear equations by using a truncated Fourier series expansion. By combining the linearization of the governing nonlinear equation with the HB method, which is termed the LHB method, analytical approximate solutions for the nonlinear oscillations of a system can be established. Unlike the classical HB method, the linearization is performed before proceeding with the harmonic balancing, which results in simple linear algebraic equations rather than nonlinear algebraic equations. The NHB method is introduced to overcome the
difficulty of achieving higher-order analytical approximations by using the classical HB method. The NHB method is similar to the LHB method, but uses Newton’s method in conjunction with the HB method to simplify the computational procedures. These approximate solutions are valid for both small and large amplitudes of oscillation, unlike the perturbation methods, which are only useful in principle for solving problems with small parameters by analytically expanding the solution in a power series of the parameter. The coefficients of the series are then obtained by solving a set of linear problems. However, in both science and engineering applications, there are many nonlinear problems that do not have small parameters, and even where such a parameter does exist, the analytical representations that are given by the perturbation methods have, in most cases, a small range of validity. Very often it is the large parameter regime of the theory under study that is of interest, and thus the small parameter requirement limits the application of the perturbation methods. To extend the scope of the available analytical approximations for solving nonlinear oscillation systems in various areas, such as weakly damped nonlinear oscillators and the two degree-of-freedom (TDOF) mass-spring system in mechanics, the implementation of supplementary methods alongside the LHB or NHB methods is attempted to improve on the results that are given by exact or numerical integration solutions.
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